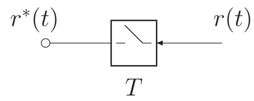
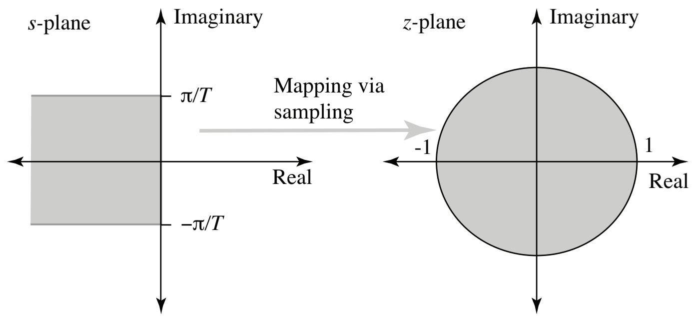
What is really happening with sampling?

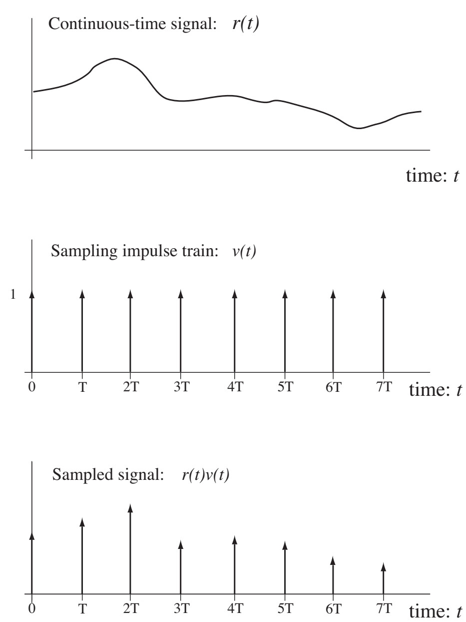


Recall that sampling maps strips of the s-plane onto the z-plane.



What does this look like from a Fourier Transform point of view?

Model of the sampling process



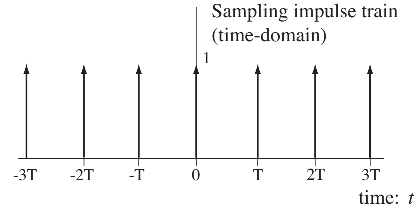
Model sampling as time-domain multiplication by a train of impulses.

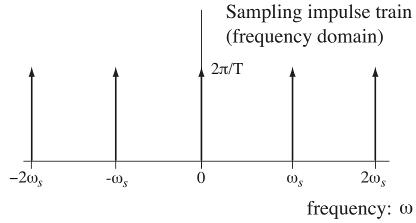
$$
\begin{aligned}
r^*(t) &= r(t)v(t) \\
&= \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT)
\end{aligned}
$$

Taking the Laplace transform,

$$
\begin{aligned}
\mathcal{L}\left\{r^*(t)\right\} &= \int_{-\infty}^{\infty} r^*(t)\mathrm{e}^{-st}dt \\
&= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT)\mathrm{e}^{-st}dt \\
&= \sum_{k=-\infty}^{\infty} r(kT)\mathrm{e}^{-skT} \\
&=: R^*(s)
\end{aligned}
$$

Impulse trains





Fourier series representation

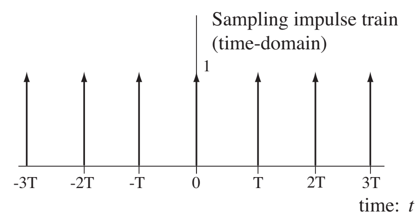
Impulse trains are periodic (with period T) and so have a Fourier series:

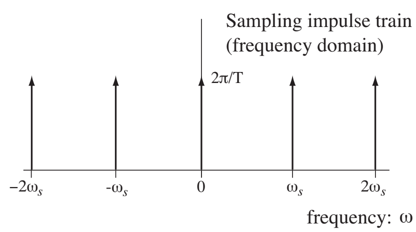
$$v(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{jn\left(\frac{2\pi}{T}\right)t}$$

The Fourier coefficients, C_n, are given by,

$$\begin{aligned} C_n &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT)\mathrm{e}^{-jn\left(\frac{2\pi}{T}\right)t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\mathrm{e}^{-jn\left(\frac{2\pi}{T}\right)t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) dt \\ &= \frac{1}{T} \end{aligned}$$

Impulse trains





Fourier series representation:

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathrm{e}^{jn\left(\frac{2\pi}{T}\right)t} \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathrm{e}^{jn\omega_s t} \end{aligned}$$

Fourier Transform:

$$v(s) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

(note: $\omega_s = 2\pi/T$).

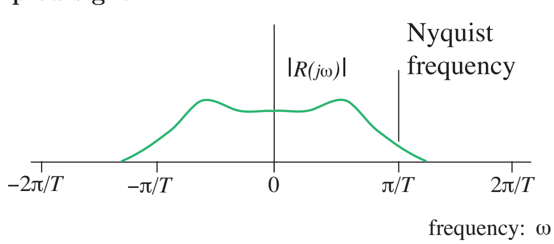
A train of impulses is equivalent to an infinite sum of (equal) sinusoids.

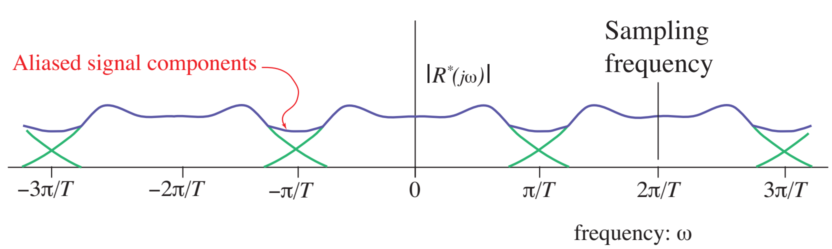
Spectrum of a sampled signal

$$
\begin{aligned}
R^*(s) = \mathcal{L}\left\{r^*(t)\right\} &= \int_{-\infty}^{\infty} r^*(t)\mathrm{e}^{-st}dt \\
&= \int_{-\infty}^{\infty} r(t)\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} \mathrm{e}^{jn\omega_s t}\right)\mathrm{e}^{-st}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)\mathrm{e}^{jn\omega_s t}\mathrm{e}^{-st}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)\mathrm{e}^{-(s-jn\omega_s)t}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s-jn\omega_s).
\end{aligned}
$$

The spectrum of the sampled signal is an infinite sum of shifted versions of the original spectrum.

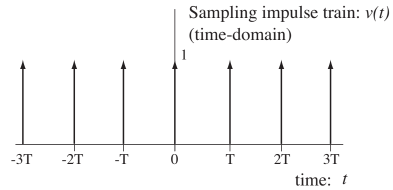
Spectrum of a sampled signal

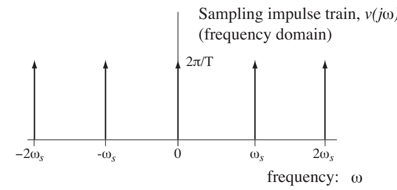




Aliasing: If $|R(j\omega)| \neq 0$ for $|w| > \pi/T$ then the shifted parts of the spectrum overlap.

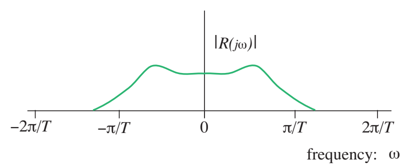
Spectrum of a sampled signal





$$v(t) \;=\; \sum_{k=-\infty}^{\infty} \delta(t-kT) \;=\; \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \quad \overset{\text{FT}}{\Longleftrightarrow} \quad v(s) \;=\; \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

Multiplication/convolution duality: $r^*(t) = r(t)v(t) \quad \overset{\text{FT}}{\Longleftrightarrow} \quad R^*(j\omega) = \frac{1}{2\pi}R(j\omega) * v(j\omega)$



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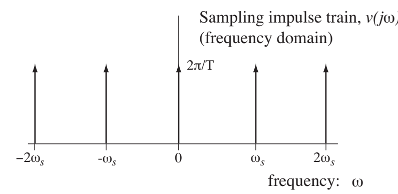
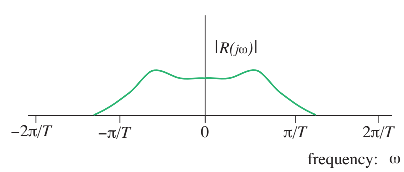
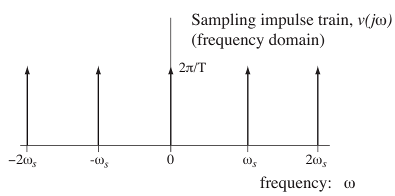


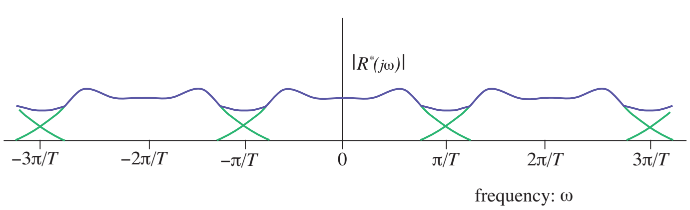
Spectrum of a sampled signal





Frequency domain convolution

$$\begin{aligned}
R^*(j\omega) &= \frac{1}{2\pi} R(j\omega) * v(j\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\nu - jn\omega_s) R(j\omega - \nu)\, d\nu \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\nu - jn\omega_s) R(j\omega - \nu)\, d\nu \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(j\omega - jn\omega_s)
\end{aligned}$$



Aliasing, what can we do?

We have seen that if $R(j\omega) \neq 0$ for $|\omega| > \pi/T$ then the signal will be aliased.

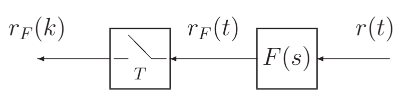
The frequency components of the original signal cannot be deduced from the sampled signal.

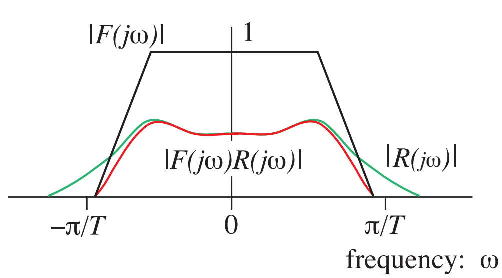
The control system will react (incorrectly) to aliased errors and disturbances.

Options

1. Sample faster. T is decreased and ω_s is increased.
   - This costs more money, and may also degrade the resolution.
2. Include a low-pass anti-aliasing filter to remove the frequency components $|\omega| > \pi/T$.
   - We lose all information about the higher frequency components.
   - In a closed-loop system the extra phase lag due to $F(s)$ degrades (or even destabilizes) the closed-loop operation.
   - Signal reconstruction is not the most critical part of closed-loop operation.

Aliasing: The effects of an anti-aliasing filter: $F(s)$



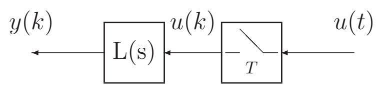


In control systems the phase effects of $F(s)$ are usually much more important (i.e. potentially destabilizing).

Reconstruction

How do we recover $r(t)$ from the sampled signal, $r^*(t)$?

Reconstruction Filter



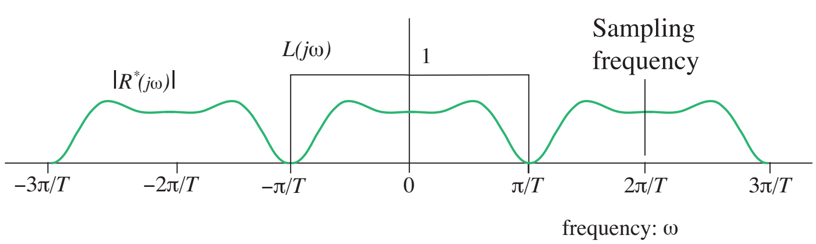
Ideal case:

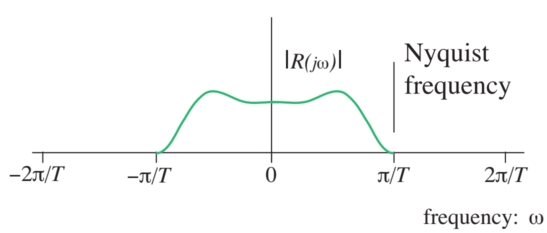
$$L(j\omega) = \begin{cases} 1, & |\omega| < \pi/T \\ 0, & |\omega| \geq \pi/T \end{cases}$$

But the impulse response of $L(s)$ is: $\dfrac{T}{\pi t}\sin(\pi t/T) = \mathrm{sinc}(\pi t/T)$

This is **acausal** and **unstable**!

Reconstruction





How good is a *ZOH* at reconstruction?

The *ZOH* is the cheapest and most readily available reconstruction filter. How good is it?

The *ZOH* frequency response (for the fundamental frequency only) is given by:

$$\begin{aligned} ZOH(j\omega) &= \frac{1 - e^{-j\omega T}}{j\omega} \\ &= e^{-j\omega T/2} \left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right) \frac{2j}{j\omega} \\ &= T e^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2} \\ &= T e^{-j\omega T/2} \text{sinc}(\omega T/2) \end{aligned}$$

So the *ZOH* looks something like a low pass filter in cascade with a delay of $T/2$ seconds.

It is not very close to the ideal filter, and it has quite a lot of phase lag.

How good is a *ZOH* at reconstruction?