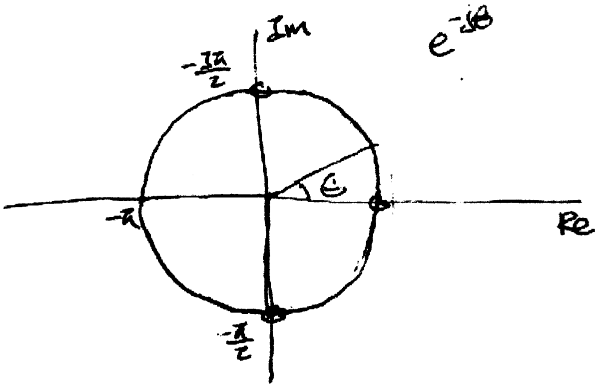
ECE 158 Fall Quarter 2007
Gibson **12/15//07**

Final Exam Solutions

1. a.) $W_N^{kN/4} = \left(e^{-j\frac{2\pi}{N}}\right)^{\frac{kN}{4}}$

$= \left(e^{-j\frac{\pi}{2}}\right)^k$

$= (-j)^k$



b.) $W_N^{kN/2} = \left(e^{-j\frac{2\pi}{N}}\right)^{\frac{kN}{2}}$

$= \left(e^{-j\pi}\right)^k$

$= (-1)^k$

c.) $W_N^{3kN/4} = \left(e^{-j\frac{2\pi}{N}}\right)^{\frac{3kN}{4}}$

$= \left(e^{-j\frac{3\pi}{2}}\right)^k$

$= (j)^k$

Problem 2.

$$X(4l+k) = \sum_{n=0}^{N-1} x(n) W_N^{(4l+k)n} \quad , 0 \le k \le 3$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{(4l+k)n} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} x(n) \cdot W_N^{(4l+k)n}$$

$$+ \sum_{n=\frac{N}{2}}^{\frac{3}{4}N-1} x(n) \cdot W_N^{(4l+k)n} + \sum_{n=\frac{3}{4}N}^{N-1} x(n) \cdot W_N^{(4l+k)n}.$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{(4l+k)n} + \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{N}{4}) \cdot W_N^{(4l+k)(n+\frac{N}{4})}$$

$$+ \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{N}{2}) W_N^{(4l+k)(n+\frac{N}{2})} + \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{3}{4}N) W_N^{(4l+k)(n+\frac{3}{4}N)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{(4l+k)n} + \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{N}{4}) W_N^{(4l+k)n + \frac{N}{4}(4l+k)}$$

$$+ \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{N}{2}) W_N^{(4l+k)n + \frac{N}{2}(4l+k)} + \sum_{n=0}^{\frac{N}{4}-1} x(n+\tfrac{3}{4}N) W^{(4l+k)(n+\frac{3}{4}N)}$$

$$W_N^{\frac{N}{4}(4l+k)} = e^{-j\frac{2\pi}{N}\frac{N}{4}(4l+k)} = e^{-j2\pi(\frac{4l+k}{4})} = e^{-j2\pi\frac{k}{4}} = (-j)^k$$

$$W_N^{\frac{N}{2}(4l+k)} = e^{-j\frac{2\pi}{N}\frac{N}{2}(4l+k)} = e^{-j2\pi\frac{k}{2}} = (-1)^k$$

$$W_N^{\frac{3}{4}N(4l+k)} = e^{-j\frac{2\pi}{N}\frac{3}{4}N(4l+k)} = e^{-j2\pi(3l+\frac{3k}{4})} = e^{-j2\pi\frac{3k}{4}} = (j)^k$$

$$X(4l) = \sum_{n=0}^{\frac{N}{4}-1} \Big(x(n) \cdot W_N^{4ln} + x(n+\tfrac{N}{4}) W_N^{4ln} \cdot 1 + x(n+\tfrac{N}{2}) W_N^{4ln}$$
$$+ x(n+\tfrac{3}{4}N) \cdot W_N^{4ln} \Big)$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left(x(n) + x(n+\tfrac{N}{4}) + x(n+\tfrac{N}{2}) + x(n+\tfrac{3}{4}N) \right) W_N^{4ln}$$

$$X(4l+1)=\sum_{n=0}^{\frac{N}{4}-1}\Big(x(n)W_N^{(4l+1)n}+x\left(n+\tfrac{N}{4}\right)W_N^{(4l+1)n}(-j)$$
$$+x\left(n+\tfrac{N}{2}\right)W_N^{(4l+1)n}(-1)+x\left(n+\tfrac{3}{4}N\right)W_N^{(4l+1)n}(j)\Big)$$
$$=\sum_{n=0}^{\frac{N}{4}-1}\left(x(n)-x\left(n+\tfrac{N}{2}\right)-j\left(x\left(n+\tfrac{N}{4}\right)-x\left(n+\tfrac{3}{4}N\right)\right)\right)W_N^{(4l+1)n}$$

$$X(4l+2)=\sum_{n=0}^{\frac{N}{4}-1}\Big(x(n)W_N^{(4l+2)n}+x\left(n+\tfrac{N}{4}\right)W_N^{(4l+2)n}(-1)+$$
$$x\left(n+\tfrac{N}{2}\right)W_N^{(4l+2)n}\cdot 1+x\left(n+\tfrac{3}{4}N\right)W_N^{(4l+2)n}(-1)\Big)$$
$$=\sum_{n=0}^{\frac{N}{4}-1}\left(x(n)+x\left(n+\tfrac{N}{2}\right)-x\left(n+\tfrac{N}{4}\right)-x\left(n+\tfrac{3}{4}N\right)\right)W_N^{(4l+2)n}$$

$$X(4l+3)=\sum_{n=0}^{\frac{N}{4}-1}\Big(x(n)W_N^{(4l+3)n}+x\left(n+\tfrac{N}{4}\right)W_N^{(4l+3)n}\cdot j+$$
$$x\left(n+\tfrac{N}{2}\right)W_N^{(4l+3)n}(-1)+x\left(n+\tfrac{3}{4}N\right)W_N^{(4l+3)n}\,j\Big)$$
$$=\sum_{n=0}^{\frac{N}{4}-1}\left(x(n)-x\left(n+\tfrac{N}{2}\right)+j\left(x\left(n+\tfrac{N}{4}\right)-x\left(n+\tfrac{3}{4}N\right)\right)\right)W_N^{(4l+3)n}$$

Problem # 3

a.) $H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-l}$

$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$.

$\hookrightarrow \quad G_1(z) = -\frac{H_0(-z)}{H_1(-z)} G_0(z) = -\frac{3-4z^{-1}}{1-2z^{-1}} G_0(z)$

$H_0(z)G_0(z) + H_1(z)G_1(z)$

$= H_0(z)G_0(z) + H_1(z)\left(-\frac{3-4z^{-1}}{1-2z^{-1}}\right) G_0(z)$

$= \left[3+4z^{-1} - \frac{3-4z^{-1}}{1-2z^{-1}}(1+2z^{-1})\right] G_0(z)$

$= \left[\frac{3-2z^{-1}-8z^{-2}-(3+2z^{-1}-8z^{-2})}{1-2z^{-1}}\right] G_0(z)$

$= \frac{-4z^{-1}}{1-2z^{-1}} G_0(z) = 2z^{-l}$

$G_0(z) = \frac{1}{2}(2z^{-1}-1)z^{-l}$

$G_1(z) = -\frac{1}{2}(2z^{-1}-1)z^{-l} \frac{3-3z^{-1}}{1-2z^{-1}}$

$= \frac{1}{2}(5-4z^{-1})z^{-l}$

b.) Show $H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-l}$

$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$

Holds.

$$(3+4z^{-1})\cdot\frac{1}{2}(2z^{-1}-1)z^{-l} + (1+2z^{-1})\cdot\frac{1}{2}(3-4z^{-1})z^{-l}$$

$$= \frac{1}{2}z^{-l}\left[(3+4z^{-1})(2z^{-1}-1) + (1+2z^{-1})(3-4z^{-1})\right]$$

$$= \frac{1}{2}z^{-l}\left[6z^{-1} \cancel{-3} + \cancel{8z^{-(l+1)}} - 4z^{-1} \cancel{+3} - 4z^{-1} + 6z^{-1} \cancel{-8z^{-(l+1)}}\right]$$

$$= \frac{1}{2}z^{-l}\cdot(12z^{-1} - 8z^{-1})$$

$$= 2z^{-l}z^{-1} \sim \text{Delay}$$

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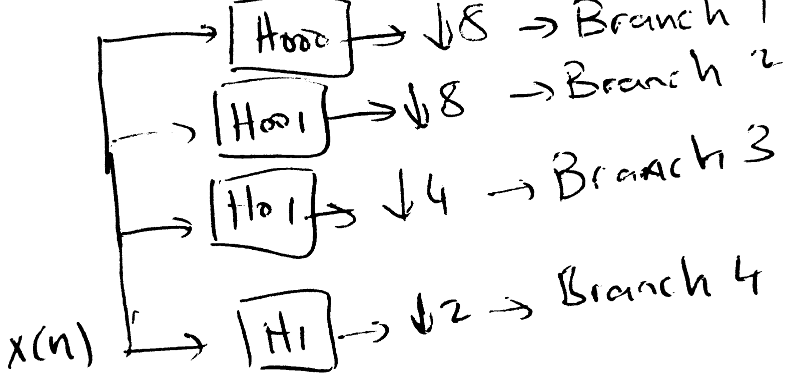


$H_1(z) = 1+2z^{-1}$

$H_{01}(z) = H_0(z).\, H_1(z^2) = (3+4z^{-1})(1+2z^{-2}) = 3+4z^{-1}+6z^{-2}+8z^{-3}$

$H_{001}(z) = H_0(z).H_0(z^2).H_1(z^4)$

$= 9+12z^{-1}+12z^{-2}+16z^{-3}+18z^{-4}+24z^{-5}+24z^{-6}+32z^{-7}$

$H_{000}(z) = H_0(z).\, H_0(z^2).\, H_0(z^4)$

$= (3+4z^{-1})(3+4z^{-2})(3+4z^{-4})$

$= 27+36z^{-1}+36z^{-2}+48z^{-3}+36z^{-4}+48z^{-5}+48z^{-6}+64z^{-7}$

Transfer function of branch 1, $T_1(z)$.

$$T_1(z) = \frac{1}{8}\sum_{k=0}^{7} H_{00}(z^{1/8} . W_8^{-k})$$

$$T_2(z) = \frac{1}{8}\sum_{k=0}^{7} H_{001}(z^{1/8} . W_8^{-k})$$

$$T_3(z) = \frac{1}{4}\sum_{k=0}^{3} H_{01}(z^{1/4} W_4^{-k})$$

$$T_4(z) = \frac{1}{2}\sum_{k=0}^{1} H_1(z^{1/2} W_2^{-k})$$

(5) A key point in solving this problem is unlike previous years exam, in this question DFT magnitudes are given. This enables you to use some relations such as zero DC term or Parseval's relation in addition to the shape of DFT and time domain signal. If DTFT was given instead of DFT, we could not use such information.

There are only 2 zero DC DFTs : (4) and (5). There are only two signals with negative component (A) and (C). (A) is obviously more high frequency where (C) is rather smooth. Hence it is easy to match (A) → (5) , (C) → (4) .

(B) is mean-added and scaled version of (E) (i.e, (E) is obtained by removing the average and scaling). Hence, its DFT should be identical to (4) except non-zero DC term, which is (6).

(A) is (H) modulated by [1, -1, 1, -1 ...] sequence, which only shifts low-freq components to highest frequency without any other change. Then, (H) → (8).

Note, (F) is downsampled and then upsampled version of (H). This operation just adds replicas in DFT domain, then (F) → (1).

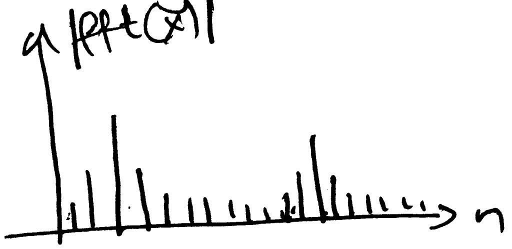
(G) is Hanning windowed version of (B). It can not be matched to (2) or (7) since they have significant high frequencies. Hence, (G) → (3) which seems to be (6) convolved with a sinc-like function.

D is rectangular windowed version of B. It should match to 2 which seems like 6 convolved with a wide sinc. Also, notice that Parseval's can be used to deduce B → 2 since energy of D is relatively higher than E, DFT preserves energy, energy of 2 is significantly higher than 7, which makes B → 2.

The remaining one is a random signal E which is matched to 7. //

Note that, there are alternative solutions that use the shape information more. For example, H is triangular, DFT should look like $\text{sinc}^2(k)$, which is 8. 6 is downsampled-upsampled version of 8, which in time domain correspond to replica B. C resembles a cosine, whose DFT resembles 4. A is a very high frequency cosine, multiplied by a low frequency cosine, which resembles 5. If F is low pass filtered, I get H ⇒ F → 1. If I sum up D and shifted D I get B, which means, DFT of D should resemble low pass filtered 6, hence D → 2.

(6) a) No, we will not see two clean impulses, since our window size is not integer number of periods, which introduces discontinuity and frequency content in nearly each bin. We'll see something like



Two impulses convolved with sinc.

b) No, windowing will not solve our problem. We have to get exactly 100 or 200 samples.

c) No, 128 samples is not integer number of periods.

d) We need to get 100 samples, then zero-pad to 128 samples. Then we can take 128 point FFT and then we'll see the clean 2 impulses.

e) Spectral leakage.