ECE 158
Gibson

Fall Quarter 2007
11/19/07

Mid-Term Exam Solutions

① a) $X(k) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi}{8} \cdot n \cdot k}$, $x(n) = [1111\,0000]$

$X(k) = 1 + e^{-j\frac{2\pi}{8} k} + e^{-j\frac{2\pi}{8} \cdot 2k} + e^{-j\frac{2\pi}{8} \cdot 3k} = \frac{1 - e^{-j\frac{2\pi}{8} \cdot 4k}}{1 - e^{-j\frac{2\pi}{8} \cdot k}}$ $(k \neq 0)$

$= \frac{1 - e^{-j\pi k}}{1 - e^{-j\frac{2\pi}{8} \cdot k}}$ $(k \neq 0)$

$X(0) = 1+1+1+1 = 4$

$X(1) = 1 - 2.4142j$

$X(2) = 0$

$X(3) = 1 - 0.4142j$

$X(4) = 0$

$X(5) = 1 + 0.4142j$

$X(6) = 0$

$X(7) = 1 + 2.4142j$

b) $Y(k) = e^{-j\frac{2\pi}{8} \cdot 2k} X(k)$

$Y(k) = \{4, -2.4142 - j, 0, 0.4142 + j, 0, 0.4142 - i, 0, -2.4142 + i\}$

② If the rows are orthogonal, the transform is an orthogonal transform.

$g_1 = (1,1,1,1)$, $g_2 = (2,1,-1,-2)$, $g_3 = (1,-1,-1,1)$, $g_4 = (1,-2,2,-1)$

$g_1 \cdot g_2^T = 0$, $g_1 \cdot g_3^T = 0$, $g_1 \cdot g_4^T = 0$, $g_2 \cdot g_4^T = 0$, $g_2\, g_3^T = 0$

$g_3 \cdot g_4^T = 0$, All rows are orthogonal, hence the transform is orthogonal

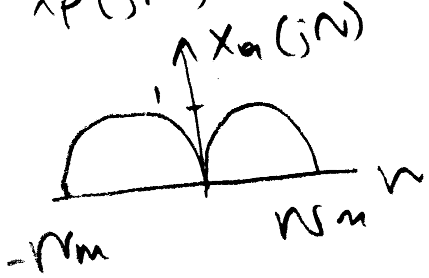
③ a) In general, before any sampling, we low-pass filter the signal to make it band-limited and hence not to cause any aliasing after sampling. Since the signal is given to be bandlimited, we can skip this step. Taking uniform samples means, multiplying the signal with impulse train. Let $x_a(t)$ denote original signal, $x_p(t)$ denote sampled signal,

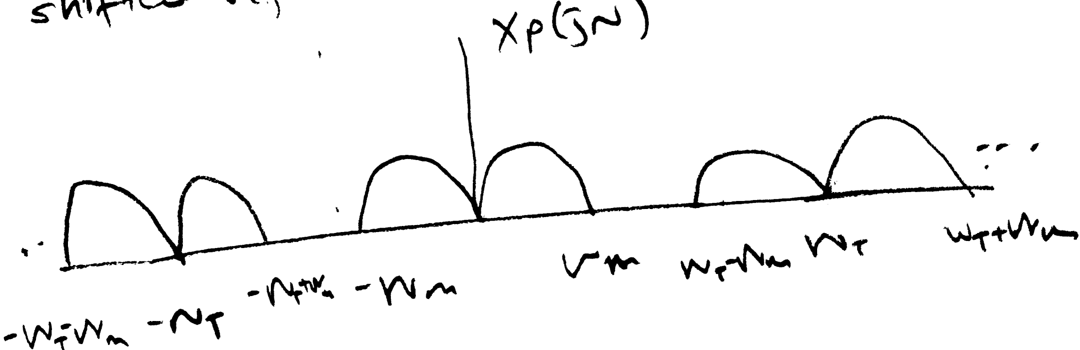
$$x_p(t) = x_a(t) \cdot p(t) \quad \text{where} \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(n - nT)$$

$T = 1/F_s$, F_s: sampling rate

$$X_p(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j\,w - k\,w_T)$$

since impulse train has a F.T as impulse train and multiplication in time domain corresponds to convolution in frequency domain. $X_p(jw)$ consists of shifted replicas at $2\pi F_s = w_T$

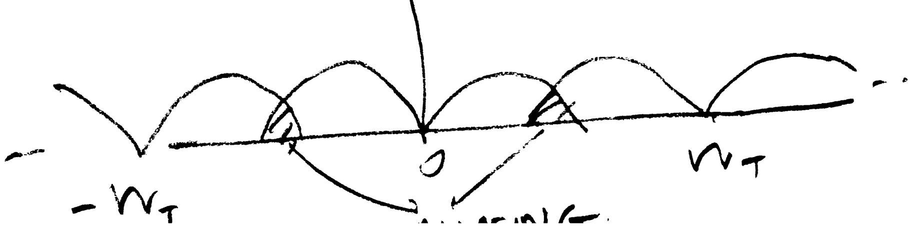




b) Aliasing occurs when replicas overlap.

$$w_T - w_m < w_m \Rightarrow \boxed{w_T < 2w_M}$$

Spectrum of aliased signal



(4) $y(n) = x(n) ④ h(n)$

$$y(n) = \sum_{k=0}^{3} x(k) \cdot h\langle\langle n-k \rangle\rangle_4 \Rightarrow y(0) = 2+4+6+2 = 14$$

$$y(1) = 4+1+8+3 = 16$$

$$y(2) = 6+2+2+4 = 14$$

$$y(3) = 8+3+4+1 = 16$$

(5) $Y(k) = H(k) \cdot G(k) = \{60, 0, -4, 0\}$

$$y(n) = IDFT\{Y(k)\} = \frac{D_4^*}{4} \cdot y(k)$$

$$y(n) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 60 \\ 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

Identical to the result of (4) as expected since

$$x(n) ④ y(n) = IDFT\{X(k) \cdot Y(k)\}$$

(6) Length of $x(n)$, $N = 5$

Length of $h(n)$, $M = 4$, $y(n) = x(n) ⑧ h(n)$

" " $y(n) = N+M-1 = 8$

We pad 3 zeros to $x(n)$, 4 zeros to $h(n)$, then take the circular convolution of zero padded $x(n)$ and $h(n)$. The result would be identical to linear convolution of $x(n)$ and $h(n)$.

(7) $G(k) = \sum_{n=0}^{N/2-1} \frac{1}{2}(x(2n) + x(2n+1))\, e^{-j\frac{2\pi}{(N/2)}.k.n}$, $\frac{N}{2}-1 \geq k \geq 0$

$H(k) = \sum_{n=0}^{N/2-1} \frac{1}{2}(x(2n) - x(2n+1))\, e^{-j\frac{2\pi}{(N/2)}.k.n}$, $\frac{N}{2}-1 \geq k \geq 0$

$$G(k) + H(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, e^{-j\frac{2\pi}{N}.k(2n)}$$

$m=2n \Rightarrow G(k) + H(k) = \sum_{\substack{m=0 \\ m=\text{even}}}^{N-2} x(m).\, e^{-j\frac{2\pi}{N}.k.m}$

$$G(k) - H(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n+1).\, e^{-j\frac{2\pi}{N}.k(2n+1)}\,.\, e^{+j\frac{2\pi}{N}.k}$$

$p = 2n+1 \Rightarrow G(k) - H(k) = \left(\sum_{\substack{p=1 \\ p:\text{odd}}}^{N-1} x(p).\, e^{-j\frac{2\pi}{N}.k.p}\right).e^{j\frac{2\pi}{N}.k}$

$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}.kn}$ (Note that: $(N-1) \geq k \geq 0$ in here)

$$X(k) = \underbrace{\sum_{\substack{m=0 \\ m:\text{even}}}^{N-2} x(m)\, e^{-j\frac{2\pi}{N}.k.m}}_{G(k)_{N/2} + H(k)_{N/2}} + \underbrace{\sum_{p=1}^{N-1} x(p)\, e^{-j\frac{2\pi}{N}.k.p}}_{[G(k)_{N/2} - H(k)_{N/2}].e^{-j\frac{2\pi}{N}.k}}$$

$$X(k) = \left(G(k)_{N/2} + H(k)_{N/2}\right) + e^{-j\frac{2\pi}{N}.k}\left[G(k)_{N/2} - H(k)_{N/2}\right]$$

We use $G(k)_{N/2}$ and $H(k)_{N/2}$ because $G(k)$ and $H(k)$ are not defined when $N-1 \geq k \geq N/2$ where $X(k)$ is defined.