

ECE 162A
Mat 162A

Lecture #12: Angular momentum
E/R: Chapter 7
F/T Chapter 10

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Classical Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

Angular momentum (Cartesian coordinates)

Classical

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

Quantum Mechanical

$$\hat{L} = \vec{r} \times \hat{p}$$

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Angular Momentum in Spherical Coordinates

$$\hat{L} = \vec{r} \times \hat{p}$$

$$\hat{L}_x = -i\hbar \left(\sin \theta \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_y = -i\hbar \left(-\cos \theta \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_z = -i\hbar \left(\frac{\partial}{\partial \phi} \right)$$

What is the z component of angular momentum?

- Calculate the expectation value

$$\bar{L}_z = \int_0^{\infty} r^2 dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \psi^* \hat{L}_z \psi$$

$$\psi = R_{nl}(r) \Theta_{lm_l} e^{im_l \phi}$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{L}_z \psi = -i\hbar \frac{\partial}{\partial \phi} e^{im_l \phi} = \hbar m_l e^{im_l \phi}$$

$$\bar{L}_z = \int_0^{\infty} R_{nl}^*(r) R_{nl}(r) r^2 dr \int_0^{\pi} \Theta_{lm_l}^* \Theta_{lm_l} d\theta \int_0^{2\pi} d\phi \hbar m_l$$

$$\bar{L}_z = \hbar m_l$$

So, the z component of angular momentum has the average value given above.

What is the total (squared) angular momentum?

- Calculate the expectation value

$$\bar{L}^2 = \int_0^{\infty} r^2 dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \psi^* \hat{L}^2 \psi$$

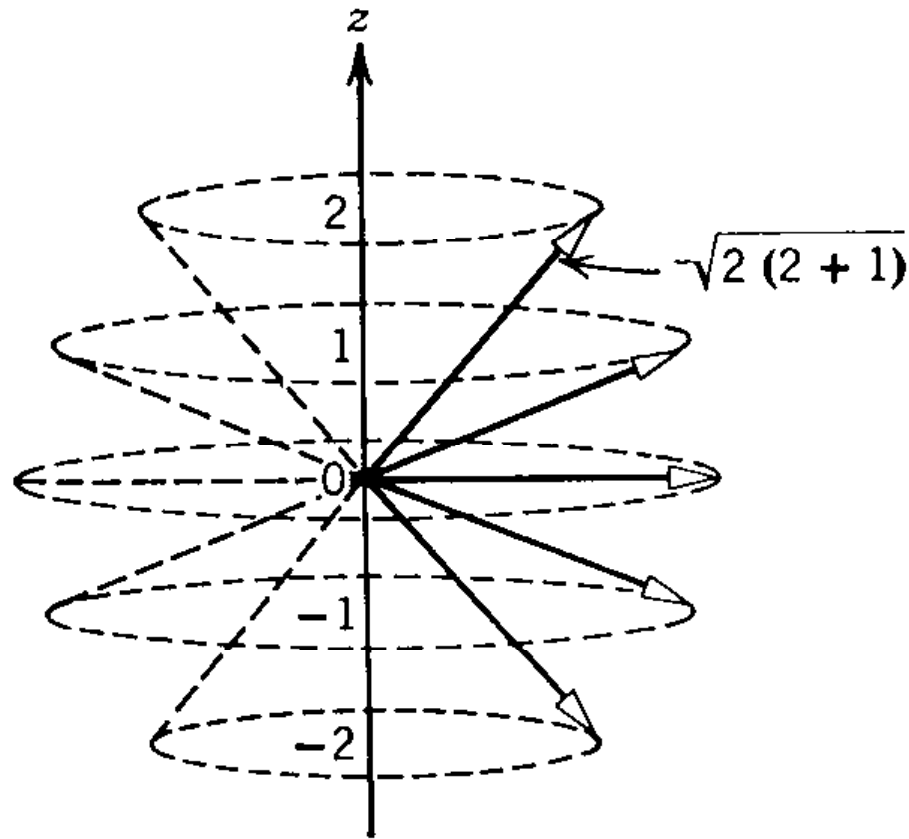
$$\psi = R_{nl}(r) \Theta_{lm_l} e^{im_l \phi}$$

$$\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\hat{L}^2 \psi = l(l+1)\hbar^2 \psi$$

$$\bar{L}^2 = l(l+1)\hbar^2$$

Vector picture of angular momentum



The arrow has length $\sqrt{2(2+1)}$

While the vertical component has length 2,1,0,-1,-2

The average value of $L_x L_y$ is zero.

The energy of the atom does not depend on m (i.e. orientation of ang. Momentum).

Quantization

- We showed that the average value of L_z is $m\hbar$. That doesn't mean that L_z is quantized.

- However, since

$$\hat{L}_z \psi = -i\hbar \frac{\partial}{\partial \phi} e^{im_1 \phi} = \hbar m_1 e^{im_1 \phi}$$

$$\bar{L}_z = \hbar m_1$$

$$\hat{L}_z^2 \psi = -\hbar^2 \frac{\partial^2}{\partial^2 \phi} e^{im_1 \phi} = \hbar^2 m_1^2 e^{im_1 \phi}$$

$$\bar{L}_z^2 = \hbar^2 m_1^2$$

- The average of a set can only equal the average of the square of the set if all values are equal. Hence, L_z is quantized.

- In general, if the quantity f has the value F in the quantum state described by ψ , then

$$\hat{f}\psi = F\psi$$

- Where \hat{f} is the operator corresponding to f .

- Note:

$$\hat{L}_x \psi \neq l_x \psi$$

$$\hat{L}_y \psi \neq l_y \psi$$

- So L_x and L_y are not quantized.

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

- Under what conditions can two or more observable properties of a quantum system have unique eigenvalues for a given quantum state?

- If two operators commute, then the eigenvalues associated with those operators are simultaneous eigenvalues

- If two operators do not commute, then the eigenvalues associated with those two operators typically exhibit an uncertainty relation.
- Exception:
- Sometimes the values are zero. For example for zero total angular momentum, $L^2=0$. $L_x=L_y=L_z=0$

- In general, for every system one may identify at least one complete set of commuting observables.

Specific Case: 2D Harmonic Oscillator

$$V(x, y) = \frac{1}{2} C(x^2 + y^2) \equiv \frac{1}{2} M\omega^2(x^2 + y^2)$$

$$\frac{-\hbar^2}{2M} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{1}{2} M\omega^2(x^2 + y^2)\psi = E\psi$$

Specific Case: 2D Harmonic Oscillator

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$$\psi(x, y) = f(x)g(y)$$

$$\frac{-\hbar^2}{2M} \left(g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} \right) + \frac{1}{2} M\omega^2(x^2 + y^2)fg = Efg$$

$$\left(\frac{-\hbar^2}{2Mf} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} M\omega^2 x^2 \right) + \left(\frac{-\hbar^2}{2Mf} \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} M\omega^2 y^2 \right) = E$$

$$\text{Cons tan } t + \text{Cons tan } t = E$$

$$\frac{-\hbar^2}{2M} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} M\omega^2 x^2 f = E_x f$$

$$\frac{-\hbar^2}{2M} \frac{\partial^2 g}{\partial y^2} + \frac{1}{2} M\omega^2 y^2 g = E_y g$$

$$E_x + E_y = E$$

F and g are just solutions of the one dimensional harmonic oscillator

$$f_n(x) = H_n\left(\frac{x}{a}\right)e^{-\frac{x^2}{a^2}}$$

With energy eigenvalue

$$E_{n_x} = \left(n_x + \frac{1}{2}\right)\hbar\omega$$

$$n_x = 0, 1, 2, \dots$$

2D Harmonic Oscillator Solutions

$$\psi_{n_x n_y} = H_{n_x} \left(\frac{x}{a} \right) H_{n_y} \left(\frac{y}{a} \right) e^{-(x^2 + y^2)/2a^2}$$

$$E = (n_x + n_y + 1)\hbar\omega$$

$$n_x = 0, 1, 2, \dots$$

$$n_y = 0, 1, 2, \dots$$

Are these solutions of \hat{L}_z ?

- Yes, if $\hat{L}_z \psi = L_z \psi$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

- We need to find linear combinations of degenerate solutions that satisfy the above equation
- Note: Degenerate solutions (solutions with the same energy) do not change in time and are called stationary solutions.

Lowest energy solution

$$n = 0 \quad n_x = n_y = 0$$

$$\psi = e^{-(x^2 + y^2)/2a^2} = e^{-r^2/2a^2}$$

$$\hat{L}_z \psi = \frac{\hbar}{i} \frac{\partial}{\partial \phi} e^{-r^2/2a^2} = 0$$

This is a solution of energy and L_z

N=1 Solutions

$$n = 1 \quad n_x = 1 \quad n_y = 0 \quad \psi_{10} = \frac{2x}{a} e^{-r^2/a^2}$$

$$n = 1 \quad n_x = 0 \quad n_y = 1 \quad \psi_{01} = \frac{2y}{a} e^{-r^2/a^2}$$

These are not solutions that satisfy:

$$\hat{L}_z \psi = L_z \psi$$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

N=1 Solutions

$$n = 1 \quad n_x = 1 \quad n_y = 0 \quad \psi_{10} = \frac{2x}{a} e^{-r^2/a^2}$$

$$n = 1 \quad n_x = 0 \quad n_y = 1 \quad \psi_{01} = \frac{2y}{a} e^{-r^2/a^2}$$

Note:

$$r e^{i\phi} = r \cos \phi + i r \sin \phi = x + iy$$

$$r e^{-i\phi} = r \cos \phi - i r \sin \phi = x - iy$$

So


$$\psi = \psi_{01} + i \psi_{10} = \frac{2(x + iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{i\phi} e^{-r^2/a^2}$$

$$\psi = \psi_{01} - i \psi_{10} = \frac{2(x - iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{-i\phi} e^{-r^2/a^2}$$

These are both solutions with $L_z = +1$ and -1 respectively.

Dirac Notation

$\psi_{n_x n_y}$ Is represented by the Dirac ket vector

$$|n_x, n_y\rangle$$


This notation is a useful shorthand:

$$|n = 1, m = 1\rangle = |1, 0\rangle + i |0, 1\rangle$$

The projection of onto all possible positions is the wave function

$$\langle x, y | n_x, n_y \rangle = \psi_{n_x n_y}$$