ECE 162A
Mat 162A

Lecture #6: Stationary Solutions
Read Chapter 5,6 of Eisberg,Resnick

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Eigenvalue Equation

- Using operators, Schroedinger's equation can be expressed as an eigenvalue equation

$$E_{op}\psi = E\psi$$

where

$$E_{op} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

- The solution of the equation involves finding the particular solutions $\psi_{n,}$ called eigenfunctions and E_n called eigenvalues.

Solutions to SE

- Free particle
- Step potential
- Infinite box
- Finite box
- Harmonic oscillator

Free particle (V=0)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$Solution: \quad \psi(x) = \exp(ikx)$$

$$where \quad \frac{\hbar^2 k^2}{2m} = E$$

The complete solution is

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} = e^{ikz-i(E/\hbar)t}$$

There are no constraints on E, any value is allowed at this point. This corresponds to a wave moving to the right. -k solutions are also valid.

Step Potential

- Sketch solutions for $E>V_0$ and $E<V_0$

Step potential

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)$$

$$Solution: \quad For\ \ E>V_0$$

$$For\ \ x<0\ \ \psi(x)=A\exp(ik_1x)+B\exp(-ik_1x)$$

$$For\ \ x>0\ \ \psi(x)=C\exp(ik_2x)+D\exp(-ik_2x)$$

$$where \quad \frac{\hbar^2 k_1{}^2}{2m}=E \quad \frac{\hbar^2 k_2{}^2}{2m}=E-V_0$$

$$Boundary\ \ conditions$$

$$\psi(x\to 0^-)=\psi(x\to 0^+)$$

$$\frac{d\psi(x\to 0^-)}{dx}=\frac{d\psi(x\to 0^+)}{dx}$$

The wave is entering from the left.

Step Potential ($E>V_0$)

$$For\ \ x<0\ \ \psi(x)=A\exp(ik_1x)+B\exp(-ik_1x)$$

$$For\ \ x>0\ \ \psi(x)=C\exp(ik_2x)+D\exp(-ik_2x)$$

$$Wave\ \ from\ \ left:\ \ D=0$$

$$\psi:\ \ A+B=C$$

$$d\psi/dx:\ \ ik_1(A-B)=ik_2C$$

$$A=\frac{1}{2}(1+\frac{k_2}{k_1})C$$

$$B=\frac{1}{2}(1-\frac{k_2}{k_1})C$$

Normalization

- C can be normalized if the density of electrons is known, or the problem is limited by either
  - A large box
  - Periodic boundary conditions
- In general, though, what matters is the reflection coefficient R and transmission coefficient T

$$R = \frac{B^* B}{A^* A}$$

$$T = \frac{C^* C}{A^* A}$$

Step Potential (E<V$_0$)

$$For \;\; x<0 \;\; \psi(x) = A\exp(ikx) + B\exp(-ikx)$$

$$For \;\; x>0 \;\; \psi(x) = C\exp(-\kappa x) + D\exp(\kappa x)$$

$$Finite \;\; wave \;\; function: \;\; D = 0$$

$$\psi: \;\; A + B = C$$

$$d\psi / dx: \;\; ik(A - B) = -\kappa C$$

$$A = \frac{1}{2}(1 + \frac{i\kappa}{k})C$$

$$B = \frac{1}{2}(1 - \frac{i\kappa}{k})C$$

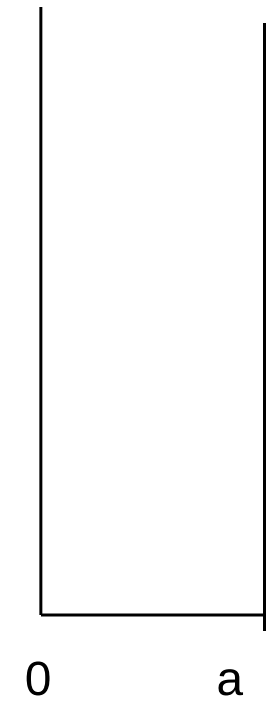
- Sketch the solution
- Explain the difference between $E>V_0$ and $E<V_0$

Problem

- Particle in an infinite box

$$V = 0 \; for \;\; 0 < x < a$$

$$V = \infty \quad otherwise$$



Solutions: $\psi(x) = A \sin kx + B \cos kx$

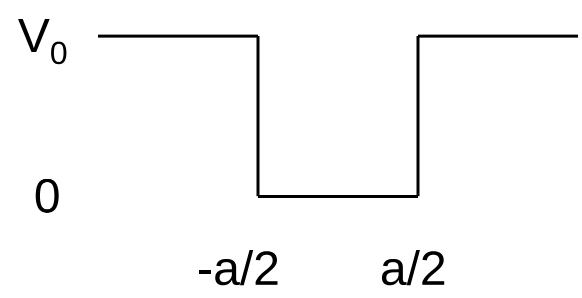
Boundary Conditions: $$\psi(0) = \psi(a) = 0$$

Eigenvalues: $$kL = n\pi \qquad n = 1,2,3,\ldots$$

$$E_n = \frac{n^2 h^2}{8ma^2}$$

Square Well

$$\frac{\hbar^2 k^2}{2m} = E \qquad \frac{\hbar^2 \kappa^2}{2m} = V_0 - E$$



$$For \;\; |x| < a/2$$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$For \;\; x < -a/2$$

$$\psi(x) = C\exp(\kappa x) + D\exp(-\kappa x)$$

$$Boundary \;\; condition: \;\; D = 0$$

$$For \;\; x > a/2$$

$$\psi(x) = F\exp(\kappa x) + G\exp(-\kappa x)$$

$$Boundary \;\; condition: \;\; F = 0$$

Solution in Appendix H

- 4 Equations (ψ and dψ/dx at two interfaces)
- 4 Unknowns (A,B,D,G)
- Solution for :

$$\varepsilon \tan \varepsilon = \sqrt{R^2 - \varepsilon^2}$$

where

$$E = \frac{2\hbar^2 \varepsilon^2}{ma^2} \qquad R^2 = \frac{mV_0 a^2}{2\hbar^2}$$