ECE 162A
Mat 162A

Lecture #12:Angular momentum
E/R: Chapter 7
F/T Chapter 10

Angular momentum (Cartesian coordinates)

Classical

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

Quantum Mechanical

$$\hat{L} = \vec{r} \times \hat{p}$$

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Angular Momentum in Spherical Coordinates

$$\hat{L} = \vec{r} \times \hat{p}$$

$$\hat{L}_x = -i\hbar(\sin\theta \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi})$$

$$\hat{L}_y = -i\hbar(-\cos\theta \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi})$$

$$\hat{L}_z = -i\hbar(\frac{\partial}{\partial\phi})$$

What is the z component of angular momentum?

- Calculate the expectation value

$$\overline{L_z} = \int_0^\infty r^2 dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \psi^* \hat{L}_z \psi$$

$$\psi = R_{nl}(r)\Theta_{lm_l} e^{im_l\phi}$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{L}_z \psi = -i\hbar \frac{\partial}{\partial \phi} e^{im_l\phi} = \hbar m_l e^{im_l\phi}$$

$$\overline{L_z} = \int_0^\infty R_{nl}*(r) R_{nl}(r) r^2 dr \int_0^\pi \Theta_{lm_l} * \Theta_{lm_l} d\theta \int_0^{2\pi} d\phi \, \hbar m_l$$

$$\overline{L_z} = \hbar m_l$$

So, the z component of angular momentum has the average value given above.

What is the total (squared) angular momentum?

- Calculate the expectation value

$$\overline{L}^2 = \int_0^\infty r^2 dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \psi^* \hat{L}^2 \psi$$

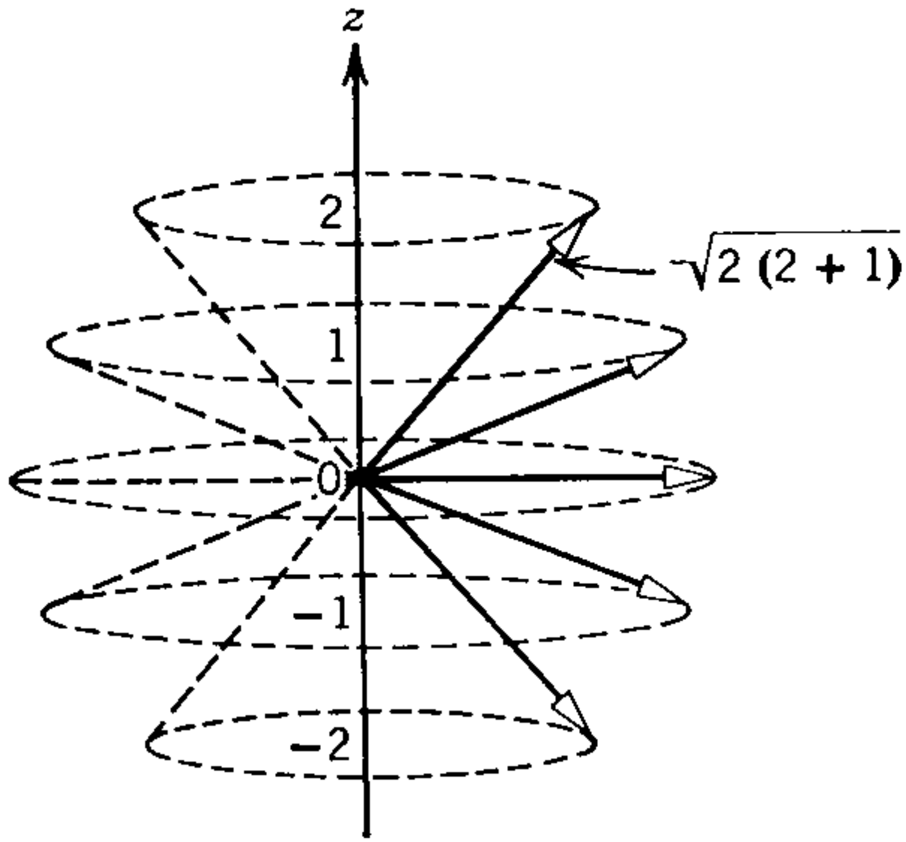
$$\psi = R_{nl}(r)\Theta_{lm_l} e^{im_l\phi}$$

$$\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right)$$

$$\hat{L}^2\psi = l(l+1)\hbar^2\psi$$

$$\overline{L}^2 = l(l+1)\hbar^2$$

Vector picture of angular momentum



The arrow has length $\sqrt{2(2+1)}$
While the vertical component has length 2,1,0,-1,-2

The average value of LxLy is zero.
The energy of the atom does not depend on m (i.e. orientation of ang. Momentum).

Quantization

- We showed that the average value of L_z is mh. That doesn't mean that L_z is quantized.
- However, since

$$\hat{L}_z\psi = -i\hbar\frac{\partial}{\partial\phi}e^{im_l\phi} = \hbar m_l e^{im_l\phi}$$

$$\overline{L_z} = \hbar m_l$$

$$\hat{L}_z^2\psi = -\hbar^2\frac{\partial^2}{\partial^2\phi}e^{im_l\phi} = \hbar^2 m_l^2 e^{im_l\phi}$$

$$\overline{L_z^2} = \hbar^2 m_l^2$$

- The average of a set can only equal the average of the square of the set if all values are equal. Hence, L_z is quantized.

- In general, if the quantity f has the value F in the quantum state described by ψ, then

$$\hat{f}\psi = F\psi$$

- Where $\hat{f}$ is the operator corresponding to f.

- Note:

$$\hat{L}_x\psi \neq l_x\psi$$

$$\hat{L}y\psi \neq l_y\psi$$

- So L_x and L_y are not quantized.

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

- Under what conditions can two or more observable properties of a quantum system have unique eigenvalues for a given quantum state?
- If two operators commute, then the eigenvalues associated with those operators are simultaneous eigenvalues

- If two operators do not commute, then the eigenvalues associated with those two operators typically exhibit an uncertainty relation.
- Exception:
- Sometimes the values are zero. For example for zero total angular momentum, $L^2=0$. $L_x=L_y=L_z=0$
- In general, for every system one may identify at least one complete set of commuting observables.

Specific Case: 2D Harmonic Oscillator

$$V(x,y) = \frac{1}{2}C(x^2 + y^2) \equiv \frac{1}{2}M\omega^2(x^2 + y^2)$$

$$\frac{-\hbar^2}{2M}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{1}{2}M\omega^2(x^2 + y^2)\psi = E\psi$$

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$$\psi(x,y) = f(x)g(y)$$

$$\frac{-\hbar^2}{2M}\left(g\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2}\right) + \frac{1}{2}M\omega^2(x^2 + y^2)f\,g = Ef\,g$$

$$\left(\frac{-\hbar^2}{2Mf}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}M\omega^2 x^2\right) + \left(\frac{-\hbar^2}{2Mf}\frac{\partial^2 f}{\partial y^2} - \frac{1}{2}M\omega^2 y^2\right) = E$$

$$Cons\tan t + Cons\tan t = E$$

$$\frac{-\hbar^2}{2M}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}M\omega^2 x^2 f = E_x f$$

$$\frac{-\hbar^2}{2M}\frac{\partial^2 g}{\partial y^2} + \frac{1}{2}M\omega^2 y^2 g = E_y g$$

$$E_x + E_y = E$$

F and g are just solutions of the one dimensional harmonic oscillator

$$f_n(x) = H_n(\frac{x}{a})e^{-\frac{x^2}{a^2}}$$

With energy eigenvalue

$$E_{n_x} = (n_x + \frac{1}{2})\hbar\omega$$

$$n_x = 0,1,2...$$

2D Harmonic Oscillator Solutions

$$\psi_{n_x n_y} = H_{n_x}(\frac{x}{a})H_{n_y}(\frac{y}{a})e^{-(x^2+y^2)/2a^2}$$

$$E = (n_x + n_y + 1)\hbar\omega$$

$$n_x = 0,1,2,...$$

$$n_y = 0,1,2,...$$

Are these solutions of $\hat{L}_z$?

- Yes, if $\hat{L}_z \psi = L_z \psi$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

- We need to find linear combinations of degenerate solutions that satisfy the above equation
- Note: Degenerate solutions (solutions with the same energy) do not change in time and are called stationary solutions.

Lowest energy solution

$$n = 0 \qquad n_x = n_y = 0$$

$$\psi = e^{-(x^2+y^2)/2a^2} = e^{-r^2/2a^2}$$

$$\hat{L}_z\psi = \frac{\hbar}{i}\frac{\partial}{\partial\phi}e^{-r^2/2a^2} = 0$$

This is a solution of energy and L_z

N=1 Solutions

$$n=1 \quad n_x=1 \quad n_y=0 \quad \psi_{10}=\frac{2x}{a}e^{-r^2/a^2}$$

$$n=1 \quad n_x=0 \quad n_y=1 \quad \psi_{01}=\frac{2y}{a}e^{-r^2/a^2}$$

These are not solutions that satisfy:

$$\hat{L}_z\psi = L_z\psi$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

N=1 Solutions

$$n = 1 \quad n_x = 1 \quad n_y = 0 \quad \psi_{10} = \frac{2x}{a} e^{-r^2/a^2}$$

$$n = 1 \quad n_x = 0 \quad n_y = 1 \quad \psi_{01} = \frac{2y}{a} e^{-r^2/a^2}$$

$Note:$

$$re^{i\phi} = r\cos\phi + ir\sin\phi = x + iy$$

$$re^{-i\phi} = r\cos\phi - ir\sin\phi = x - iy$$

So

$$\psi = \psi_{01} + i\psi_{01} = \frac{2(x+iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{i\phi} e^{-r^2/a^2}$$

$$\psi = \psi_{01} - i\psi_{01} = \frac{2(x-iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{-i\phi} e^{-r^2/a^2}$$

These are both solutions with L_z= +1 and -1 respectively.

Dirac Notation

$\psi_{n_x n_y}$ Is represented by the Dirac ket vector

$| n_x , n_y >$

This notation is a useful shorthand:

$$| n = 1, m = 1 >=| 1,0 > + i \,| 0,1 >$$

The projection of onto all possible positions is the wave function

$$< x, y \,| n_x , n_y >= \psi_{n_x n_y}$$