Lecture #12: Angular momentum
E/R: Chapter 7
F/T Chapter 10
Angular momentum
(Classical, Quantum Mechanical)

\[ \vec{L} = \vec{r} \times \vec{p} \]
\[ L_x = yp_z - zp_y \]
\[ L_y = zp_x - xp_z \]
\[ L_z = xp_y - yp_x \]
\[ \hat{L} = \hat{\vec{r}} \times \hat{\vec{p}} \]
\[ \hat{L}_x = -i\hbar(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) \]
\[ \hat{L}_y = -i\hbar(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}) \]
\[ \hat{L}_z = -i\hbar(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) \]
Angular Momentum in Spherical Coordinates

\[ \hat{L} = \hat{r} \times \hat{p} \]

\[ \hat{L}_x = -i\hbar (\sin \theta \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} ) \]

\[ \hat{L}_y = -i\hbar (-\cos \theta \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} ) \]

\[ \hat{L}_z = -i\hbar \left( \frac{\partial}{\partial \phi} \right) \]
What is the z component of angular momentum?

- Calculate the expectation value

\[
\overline{L}_z = \int_0^\infty R_{nl}(r) \Theta_{lm} e^{im\phi} \left( -i\hbar \frac{\partial}{\partial \phi} \right) R_{nl}(r) \Theta_{lm} e^{im\phi} d\phi \\
= \int_0^{2\pi} \Theta_{lm} \Theta_{lm} \hbar m_l d\phi
\]

So, the z component of angular momentum has the average value given above.
What is the total (squared) angular momentum?

- Calculate the expectation value

\[ \overline{L^2} = \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} r^2 dr \, d\theta \, d\phi \, \psi^* \hat{L}^2 \psi \]

\[ \psi = R_{nl}(r) \Theta_{lm} e^{im\phi} \]

\[ \hat{L}^2 = -\hbar^2 \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial^2 \phi} \right) \]

\[ \hat{L}^2 \psi = l(l + 1)\hbar^2 \psi \]

\[ \overline{L^2} = l(l + 1)\hbar^2 \]
Vector picture of angular momentum

The arrow has length $\sqrt{2(2+1)}$
While the vertical component has length 2,1,0,-1,-2

The average value of $L_xL_y$ is zero.
The energy of the atom does not depend on $m$ (i.e. orientation of ang. Momentum).
Quantization

• We showed that the average value of $L_z$ is $m_h$. That doesn’t mean that $L_z$ is quantized.

• However, since

$$\hat{L}_z \psi = -i\hbar \frac{\partial}{\partial \phi} e^{im_l \phi} = \hbar m_l e^{im_l \phi}$$

$$\overline{L}_z = \hbar m_l$$

$$\hat{L}_z^2 \psi = -\hbar^2 \frac{\partial^2}{\partial^2 \phi} e^{im_l \phi} = \hbar^2 m_l^2 e^{im_l \phi}$$

$$\overline{L}_z^2 = \hbar^2 m_l^2$$

• The average of a set can only equal the average of the square of the set if all values are equal. Hence, $L_z$ is quantized.
• In general, if the quantity $f$ has the value $F$ in the quantum state described by $\psi$, then

$$\hat{f}\psi = F\psi$$

• Where $\hat{f}$ is the operator corresponding to $f$. 
• Note:

\[ \hat{L}_x \psi \neq l_x \psi \]

\[ \hat{L}_y \psi \neq l_y \psi \]

• So \( L_x \) and \( L_y \) are not quantized.
\[ [L_x, L_y] = i\hbar L_z \]
\[ [L_y, L_z] = i\hbar L_x \]
\[ [L_z, L_x] = i\hbar L_y \]
• Under what conditions can two or more observable properties of a quantum system have unique eigenvalues for a given quantum state?

• If two operators commute, then the eigenvalues associated with those operators are simultaneous eigenvalues
• If two operators do not commute, then the eigenvalues associated with those two operators typically exhibit an uncertainty relation.

• Exception:

• Sometimes the values are zero. For example for zero total angular momentum, $L^2=0$. $L_x=L_y=L_z=0$

• In general, for every system one may identify at least one complete set of commuting observables.
Specific Case: 2D Harmonic Oscillator

\[ V(x, y) = \frac{1}{2} C(x^2 + y^2) = \frac{1}{2} M\omega^2 (x^2 + y^2) \]

\[ -\hbar^2 \frac{\partial^2 \psi}{2M \partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{2} M\omega^2 (x^2 + y^2) \psi = E \psi \]
Specific Case: 2D Harmonic Oscillator

\[ V(x, y) = \frac{1}{2} C(x^2 + y^2) \equiv \frac{1}{2} M\omega^2 (x^2 + y^2) \]

\[ -\frac{\hbar^2}{2M} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{1}{2} M\omega^2 (x^2 + y^2) \psi = E\psi \]

\[ \psi (x, y) = f(x)g(y) \]

\[ -\frac{\hbar^2}{2M} \left( g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} \right) + \frac{1}{2} M\omega^2 (x^2 + y^2) f \ g = E \ f \ g \]

\[ \left( -\frac{\hbar^2}{2Mf} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} M\omega^2 x^2 \right) + \left( -\frac{\hbar^2}{2Mf} \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} M\omega^2 y^2 \right) = E \]

\[ \text{Cons tan } t + \text{Cons tan } t = E \]

\[ -\frac{\hbar^2}{2M} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} M\omega^2 x^2 \ f = E_x f \]

\[ -\frac{\hbar^2}{2M} \frac{\partial^2 g}{\partial y^2} + \frac{1}{2} M\omega^2 y^2 \ g = E_y g \]

\[ E_x + E_y = E \]

ECE/Mat 162A, Blumenthal, Fall 2009
F and g are just solutions of the one dimensional harmonic oscillator

\[ f_n(x) = H_n\left(\frac{x}{a}\right)e^{-\frac{x^2}{a^2}} \]

With energy eigenvalue

\[ E_{n_x} = \left(n_x + \frac{1}{2}\right)\hbar \omega \]

\[ n_x = 0,1,2,\ldots \]
2D Harmonic Oscillator Solutions

\[ \psi_{n_x n_y} = H_{n_x} \left( \frac{x}{a} \right) H_{n_y} \left( \frac{y}{a} \right) e^{-\left(\frac{x^2 + y^2}{2a^2}\right)} \]

\[ E = (n_x + n_y + 1) \hbar \omega \]

\[ n_x = 0, 1, 2, \ldots \]

\[ n_y = 0, 1, 2, \ldots \]
Are these solutions of $^\wedge L_z$?

- Yes, if $\hat{L}_z \psi = L_z \psi$

\[
\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}
\]

- We need to find linear combinations of degenerate solutions that satisfy the above equation.
- Note: Degenerate solutions (solutions with the same energy) do not change in time and are called stationary solutions.
Lowest energy solution

\[ n = 0 \quad n_x = n_y = 0 \]

\[ \psi = e^{-(x^2 + y^2)/2a^2} = e^{-r^2/2a^2} \]

\[ \hat{L}_z \psi = \frac{\hbar}{i} \frac{\partial}{\partial \phi} e^{-r^2/2a^2} = 0 \]

This is a solution of energy and \( L_z \)
N=1 Solutions

\[ n = 1 \quad n_x = 1 \quad n_y = 0 \quad \psi_{10} = \frac{2x}{a} e^{-r^2/a^2} \]

\[ n = 1 \quad n_x = 0 \quad n_y = 1 \quad \psi_{01} = \frac{2y}{a} e^{-r^2/a^2} \]

These are not solutions that satisfy:

\[ \hat{L}_z \psi = L_z \psi \]

\[ \hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \]
N=1 Solutions

\[ n = 1 \quad n_x = 1 \quad n_y = 0 \quad \psi_{10} = \frac{2x}{a} e^{-r^2/a^2} \]

\[ n = 1 \quad n_x = 0 \quad n_y = 1 \quad \psi_{01} = \frac{2y}{a} e^{-r^2/a^2} \]

Note:

\[ re^{i\phi} = r \cos \phi + ir \sin \phi = x + iy \]

\[ re^{-i\phi} = r \cos \phi - ir \sin \phi = x - iy \]

So

\[ \psi = \psi_{01} + i\psi_{01} = \frac{2(x + iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{i\phi} e^{-r^2/a^2} \]

\[ \psi = \psi_{01} - i\psi_{01} = \frac{2(x - iy)}{a} e^{-r^2/a^2} = \frac{2r}{a} e^{-i\phi} e^{-r^2/a^2} \]

These are both solutions with \( L_z = +1 \) and \(-1\) respectively.
Dirac Notation

\[ \psi_{n_xn_y} \] is represented by the Dirac ket vector

\[ | n_x, n_y \rangle \]

This notation is a useful shorthand:

\[ | n = 1, m = 1 \rangle = | 1, 0 \rangle + i | 0, 1 \rangle \]

The projection of onto all possible positions is the wave function

\[ < x, y | n_x, n_y >= \psi_{n_xn_y} \]