ECE162A
Homework #4

Problem 1

The Schrödinger equation must be linear in the wave function to adhere to the principle of superposition, which is only valid for linear systems. The principle of superposition states that sum of basis wavefunctions that are solutions to the Schrödinger equation results in another wavefunction, which is also a solution. ■

Problem 2

We know that classical wave equations contain a second space derivative and a second time derivative, while the Schrödinger equation contains a second space derivative and a first time derivative. If we look at the time derivative aspect of each equation with a simple propagating wave:

$$\Psi(x,t) \propto \exp\left(i(kz - \omega t)\right)$$

We see that:

$$\begin{aligned} \frac{d\Psi(x,t)}{dt} &= -i\omega \exp\left(i(kz - \omega t)\right) \\ \frac{d^2\Psi(x,t)}{dt^2} &= \omega^2 \exp\left(i(kz - \omega t)\right) \end{aligned}$$

■

Also, the use of complex notation in the Schrödinger equation requires that $\Psi(x,t)$ be a complex entity. This notation originates from the first derivative relationship with time. With classical waves, we take the real part of $\Psi(x,t)$ to be the "actual" wave. With solutions to the Schrödinger equation, the real and complex part are both used when calculating expectation values, propagation, etc.

Problem 3

These solutions refer to Figure 5-20 in Eisberg and Resnick.

Part A Since the probability of finding the particle within dx is:

$$\int \Psi(x,t)^* \Psi(x,t) dx$$

we can see that the maximum probability is at $x = -1$ for the time t shown in the plot.

Part B The probability of finding the particle is small when $\Psi(x,t)$ is small. From this, we can show that the particle is least likely to be found at $x = 0$ and $x \rightarrow \pm\infty$, where $\Psi(x,t) \rightarrow 0$.

Part C By inspecting the plot, we can see that:

$$\int_{-\infty}^{0} \Psi(x,t)^*\Psi(x,t)dx < \int_{0}^{\infty} \Psi(x,t)^*\Psi(x,t)dx$$

Therefore, the particle is most likely to be found at positive values for x. ■

Part D By inspection, we see that $\Psi(x,t)$ approaches 0 as $x \to \pm\infty$. This indicates that there is a *finite* potential step at some positive and negative value of x, confining the particle. The decay rate appears to be slower (smaller α) for positive values of x, and since:

$$\alpha = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

we can say that $V - E$ for the step at positive x is less than the step at negative x. One way to estimate $V(x)$ between regions where $\Psi(x,t)$ decays is to approximate curves that peak at $x = -1$ and $x \approx 3.5$ with sinusoidal functions. If we separate $\Psi(x,t)$ into two sinusoidal functions with different values for k, we get:

$$\Psi(x,t) = \begin{cases} 5\sin\left(\frac{\pi x}{2}\right) & \text{for} \quad 0 < x < x|_{step,x} \\ 3\sin\left(\frac{\pi x}{7}\right) & \text{for} \quad x|_{step,-x} < x < 0 \end{cases} \tag{1}$$

Since we have a fixed t, we can use the time-independent Schrödinger equation and show that:

$$\frac{d^2\psi(x)}{dx^2}\frac{1}{\psi(x)} \propto V(x) - E$$

If we substitute the expressions from equation 1 into the relationship given above, we have:

$$(V-E)|_{+x} = \frac{-\pi^2}{4}$$
$$(V-E)|_{-x} = \frac{-\pi^2}{49}$$

or, in general, E is greater than V between the step at negative x and 0, while the difference between E and V is a factor of ≈ 12 smaller between $x = 0$ and the step at positive x. A sketch is shown in Figure 1.

This is one method of approximating $V(x)$ and becomes more accurate for bound particles with larger energy. There are several others ways to approach Part D.

Part E The wave function corresponds to the energy level at $n = 2$, since it is antisymmetric with one node.

Problem 4

From Example 5-9 and 5-10, we have:

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{a}\right)\exp\left(\frac{-iEt}{\hbar}\right)$$

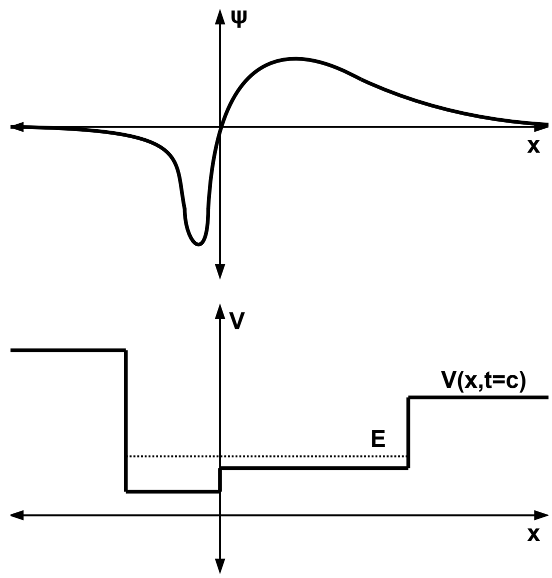

Figure 1: Sketch of $V(x)$ for $\Psi(x,t)$ given in Problem 2.

Part A The probability of finding the particle a distance of $a/3$ to the right-hand end of the box of length a is:

$$
\begin{aligned}
P(a/6 < x < a/2) &= \int_{a/6}^{a/2} \Psi(x,t)^* \Psi(x,t) dx \\
&= \frac{2}{a} \int_{a/6}^{a/2} \cos^2\left(\frac{\pi x}{a}\right) dx \\
&= \frac{1}{a} \int_{a/6}^{a/2} \left[1 + \cos\left(\frac{2\pi x}{a}\right)\right] dx \\
&= \frac{1}{a} \left[x + \frac{a}{2\pi} \sin\left(\frac{2\pi x}{a}\right)\right]_{a/6}^{a/2} \\
&= \frac{1}{a} \left[\frac{a}{3} - \frac{a\sqrt{3}}{4\pi}\right] \\
&= 0.1955
\end{aligned}
$$

Part B Classically, we can say that the probability is uniform between $-a/2$ and $a/2$, so the probability between $a/6$ and $a/2$ is:

$$
\begin{aligned}
P(a/6 < x < a/2) &= \frac{a/2 - a/6}{a} \\
&= \frac{a/3}{a} \\
&= \frac{1}{3}
\end{aligned}
$$

■

Problem 5

Given:

$$\Psi(x,t) = \begin{cases} A \sin\left(\frac{2\pi x}{a}\right) \exp\left(\frac{-iEt}{\hbar}\right) & \text{for} \quad -a/2 < x < a/2 \\ 0 & \text{for} \quad |x| > a/2 \end{cases}$$

Part A and B If we set $V = 0$, the Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x,t)}{dx^2} = i\hbar\frac{d\Psi(x,t)}{dt}$$

Plugging $\Psi(x,t)$ into the SE:

$$-\frac{\hbar^2}{2m}\frac{-4\pi^2}{a^2}\Psi(x,t) = i\hbar\frac{-iE}{\hbar}Psi(x,t)$$
$$\frac{4\pi^2\hbar^2}{2ma^2}\Psi(x,t) = E\Psi(x,t)$$

Solving for energy E gives:

$$E = \frac{4\pi^2\hbar^2}{2ma^2}$$

We can see that this is the equation for the energy of a confined particle with with $n = 2$. The energy for a particle in the ground state is given by:

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

so $E_2 = 4E_1$.

Part C The space dependence of the wave functions is plotted in Figure 2. These functions are:

$$\psi_1(x) = A\cos\left(\frac{\pi x}{a}\right)$$
$$\psi_2(x) = A\sin\left(\frac{2\pi x}{a}\right)$$

From the wave functions, we can see that $\left|d^2\psi_2(x)/dx^2\right| > \left|d^2\psi_1(x)/dx^2\right|$ for most x. From the time-independent equation, we know that:

$$\frac{d^2\psi(x)}{x^2} = \frac{2m}{\hbar^2}\left[V(x) - E\right]\psi(x)$$

For $V(x) = 0$, this simplifies to:

$$-\frac{d^2\psi(x)}{x^2} = \frac{2mE}{\hbar^2}\psi(x)$$

Therefore, $E_2 > E_1$ since

$$\left|\frac{d^2\psi_2(x)}{dx^2}\right| > \left|\frac{d^2\psi_1(x)}{dx^2}\right|$$
$$\frac{2mE_2}{\hbar^2}\psi_2(x) > \frac{2mE_1}{\hbar^2}\psi_1(x)$$
$$E_2 > E_1$$

and both $\psi_1(x)$ and $\psi_2(x) \leq A$. This is true for most values of x. ■

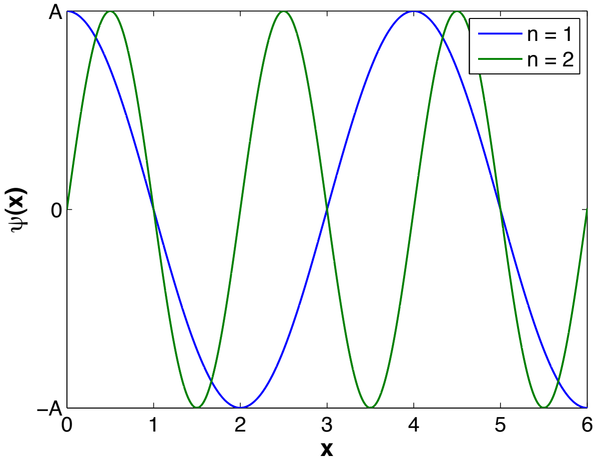

Figure 2: $\Psi(x,t)$ with fixed t for $n = 1, 2$ and a = 2.

Problem 6

Normalizing the wave function.

Part A The wave equation can be normalized by solving for A that satisfies the following expression:

$$\int_{-a/2}^{a/2} A^2 \cos^2\left(\frac{2\pi x}{a}\right) dx = 1$$

We then evaluate the integral to find A:

$$\begin{aligned} \int_{-a/2}^{a/2} A^2 \sin^2\left(\frac{2\pi x}{a}\right) dx &= 1 \\ \frac{A^2}{2}\int_{a/2}^{a/2} \left[1 + \sin\left(\frac{4\pi x}{a}\right)\right] dx &= 1 \\ \frac{A^2}{2}\left[x + \frac{a}{4\pi}\sin\left(\frac{4\pi x}{a}\right)\right]_{-a/2}^{a/2} &= 1 \\ \frac{A^2}{2}a &= 1 \end{aligned}$$

or,

$$A = \sqrt{\frac{2}{a}}$$

■

The normalized wave equation is then:

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\sin^2\left(\frac{2\pi x}{a}\right)\exp\left(\frac{-iEt}{\hbar}\right) \quad \text{for } -a/2 < x < a/2$$

and $\Psi(x,t) = 0$ otherwise.

Part B The value of A obtained for the normalized ground state is the same as the wave function in this problem. This is due to the wave function being periodic with simple sines or cosines.