In this problem, we have a particle traveling in the $-x$ direction and incident on a step located at $x = 0$. The particle is initially in the region of $x > 0$ where the potential $V(x) = V_0$ and the step represents a potential drop to $V(x) = 0$ for $x < 0$.

We can keep the coefficients of the wave equations in each region to be consistent with what we have previously defined. The equation for the region where $x > 0$ is:

$$
\psi_{II}(x) = A \exp(-i k_{II} x) + B \exp(i k_{II} x) \quad \text{where} \quad k_{II} = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}
$$

where $A$ is the coefficient for the wave traveling towards $-x$ and $B$ is the reflected wave in the $+x$ direction. The equation for the region where $x < 0$ is:

$$
\psi_I(x) = C \exp(-i k_I x) \quad \text{where} \quad k_I = \sqrt{\frac{2mE}{\hbar^2}}
$$

where $C$ is the coefficient for the transmitted wave at the potential step. We can eliminate $D$ since the particle can not be transmitted across the step from $-x$ to $+x$.

Utilizing boundary conditions for the continuity of $d\psi(x)/dx$ and $\psi(x)$ across the step at $x = 0$, we obtain:

$$
A + B = C
$$

$$
-k_{II}A + k_{II}B = -k_I C
$$

Since the objective is to obtain reflection ($R$) and transmission ($T$) coefficients, we can rearrange the two boundary condition equations to find $B/A$ and $C/A$. These are:

$$
\left( \frac{B}{A} \right) = \frac{k_{II} - k_I}{k_{II} + k_I} \quad \text{(1)}
$$

$$
\left( \frac{C}{A} \right) = \frac{2k_{II}}{k_{II} + k_I} \quad \text{(2)}
$$

From Section 6-4, we know that $R$ and $T$ can be expressed as:

$$
R = \frac{B^*B}{A^*A}
$$

$$
T = \frac{\nu_I C^*C}{\nu_{II} A^*A}
$$
where \(v_I\) and \(v_{II}\) have been inverted from Eq 6-41 due to the change in particle direction. Plugging in our values from Equations 1 and 2 and letting \(v_i = \hbar k_i/m\), \(R\) and \(T\) can be written as:

\[
R = \frac{B^2}{A^2} = \left( \frac{k_{II} - k_I}{k_{II} + k_I} \right)^2 \\
T = \left( \frac{\hbar k_I/m}{\hbar k_{II}/m} \right) \left( \frac{2k_{II}}{k_{II} + k_I} \right)^2 \\
    = \frac{4k_Ik_{II}^2}{k_{II}(k_{II} + k_I)^2} \\
    = \frac{4k_Ik_{II}}{(k_I + k_{II})^2}
\]

The equations for \(R\) and \(T\) are identical to those found in Equations 6-40 and 6-42.

---

**E&R - Ch. 6, Problem 7**

Since this problem is a modification of Problem 5 (\(E > V_o\) compared to \(E < V_o\)), the algebra that results in the tunneling equation did not have to be shown. This was not clear, and the work is shown below for those who are interested.

**Part A** The potential for this problem is:

\[
V(x) = \begin{cases} 
0 & \text{for } x < 0 \\
V_o & \text{for } 0 < x < a \\
0 & \text{for } x > a
\end{cases}
\]

For the general solution, regions I, II, and III correspond to \(x < 0\), \(0 < x < a\), and \(x > a\) respectively. The particle is also assumed to be traveling in the \(+x\) direction with \(E > V_o\). The general solution is then:

\[
\psi_I(x) = A \exp (ik_Ix) + B \exp (-ik_Ix) \\
\psi_{II}(x) = F \exp (ik_{II}x) + G \exp (-ik_{II}x) \\
\psi_{III}(x) = C \exp (ik_Ix)
\]

where

\[
k_I = \sqrt{\frac{2mE}{\hbar^2}} \\
k_{II} = \sqrt{\frac{2m(E - V_o)}{\hbar^2}}
\]

and \(D = 0\) since the particle can not travel from \(+x\) to \(-x\) across the step at \(x = a\). Utilizing boundary conditions for the continuity of \(\psi(x)\) across the step at \(x = 0\) and \(x = a\), we obtain:

\[
A + B = F + G \\
F \exp (ik_{II}a) + G \exp (-ik_{II}a) = C \exp (ik_Ia)
\]
Similarly, boundary conditions for the continuity of \(d\psi(x)/dx\) across the step at \(x = 0\) and \(x = a\) results in:

\[
k_1 A - k_1 B = k_{111} F - k_{111} G
\]
\[
k_{111} F \exp (ik_{111} a) - k_{111} G \exp (-ik_{111} a) = k_1 C \exp (ik_1 a)
\]

We can now start isolating variables to make these equations a little easier to work with. From the first boundary condition equations, we can write:

\[
A + B - G = F
\]

and

\[
(A + B - G) \exp (ik_{111} a) + G \exp (-ik_{111} a) = C \exp (ik_1 a) \tag{3}
\]
\[
\rightarrow A \exp (ik_{111} a) + B \exp (ik_{111} a) - G [\exp (ik_{111} a) - \exp (-ik_{111} a)] = C \exp (ik_1 a) \tag{4}
\]

Performing the same substitution with the second set of boundary conditions results in:

\[
k_1 A - k_1 B = k_{111} (A + B - G) - k_{111} G \tag{5}
\]
\[
\rightarrow A (k_1 - k_{111}) - B (k_1 + k_{111}) = -2k_{111} G \tag{6}
\]

and

\[
k_{111} (A + B - G) \exp (ik_{111} a) - k_{111} G \exp (-ik_{111} a) = k_1 C \exp (ik_1 a) \tag{7}
\]
\[
\rightarrow k_{111} A \exp (ik_{111} a) + k_{111} B \exp (ik_{111} a) - k_{111} G [\exp (-ik_{111} a) + \exp (ik_{111} a)] = k_1 C \exp (ik_1 a) \tag{8}
\]

From Equation 6, we can write:

\[
G = -\frac{1}{2k_{111}} [A (ik_1 - ik_{111}) - B (ik_1 + ik_{111})]
\]

We can then substitute this expression into Equation 4:

\[
A \exp (ik_{111} a) + B \exp (ik_{111} a) + \frac{1}{2k_{111}} [A (ik_1 - ik_{111}) - B (ik_1 + ik_{111})] ... \tag{9}
\]
\[
[\exp (-ik_{111} a) - \exp (ik_{111} a)] = C \exp (ik_1 a) \tag{10}
\]
\[
\rightarrow A \left[ \left( \frac{1}{2} - \frac{k_1}{2k_{111}} \right) \exp (-ik_{111} a) + \left( \frac{1}{2} + \frac{k_1}{2k_{111}} \right) \exp (ik_{111} a) \right] + ... \tag{11}
\]
\[
B \left[ \left( \frac{1}{2} + \frac{k_1}{2k_{111}} \right) \exp (-ik_{111} a) + \left( \frac{1}{2} - \frac{k_1}{2k_{111}} \right) \exp (ik_{111} a) \right] = C \exp (ik_1 a) \tag{12}
\]

Similarly, substituting the expression for \(G\) into Equation 8 gives:

\[
k_{111} A \exp (ik_{111} a) + k_{111} B \exp (ik_{111} a) + \frac{1}{2} [A (ik_1 - ik_{111}) - B (ik_1 + ik_{111})] ... \tag{13}
\]
\[
[\exp (-ik_{111} a) + \exp (ik_{111} a)] = k_1 C \exp (ik_1 a) \tag{14}
\]
\[
\rightarrow A \left[ \left( \frac{1}{2} - \frac{k_1}{2k_{111}} \right) \exp (-ik_{111} a) - \left( \frac{1}{2} + \frac{k_1}{2k_{111}} \right) \exp (ik_{111} a) \right] + ... \tag{15}
\]
\[
B \left[ \left( \frac{1}{2} + \frac{k_1}{2k_{111}} \right) \exp (-ik_{111} a) - \left( \frac{1}{2} - \frac{k_1}{2k_{111}} \right) \exp (ik_{111} a) \right] = C \frac{k_1}{k_{111}} \exp (ik_1 a) \tag{16}
\]
Now that we have two equations with three unknowns, we can solve for coefficients and make substitutions to find the tunneling coefficient. If we rewrite Equation 12 in terms of $B$ and define two terms ($p$ and $q$) to aid in the algebra, we have:

\[
p = 1 + \frac{k_I}{k_{111}}
\]

\[
q = 1 - \frac{k_I}{k_{111}}
\]

\[
\therefore B = \frac{C \exp(ik_Ia) - \frac{A}{2} [q \exp(-ik_{111}a) + p \exp(ik_{111}a)]}{p \exp(-ik_{111}a) + q \exp(ik_{111}a)}
\]

Substituting $B$ into Equation 16 and rearranging to express $A$ in terms of $C$ results in:

\[
\frac{A}{2} [q \exp(-ik_{111}a) - p \exp(ik_{111}a)] = C \frac{k_I}{k_{111}} \exp(ik_Ia)
\]

\[
\rightarrow \frac{A}{2} (q \exp(-ik_{111}a) - p \exp(ik_{111}a)) = A \frac{p \exp(-ik_{111}a) - q \exp(ik_{111}a)}{2 \exp(-ik_{111}a) + q \exp(ik_{111}a)} \times ...
\]

\[
(q \exp(-ik_{111}a) + p \exp(ik_{111}a)) = C \left( \frac{k_I}{k_{111}} - \frac{p \exp(-ik_{111}a) - q \exp(ik_{111}a)}{2 \exp(-ik_{111}a) + q \exp(ik_{111}a)} \right) \exp(ik_Ia)
\]

We can simplify this expression even more by multiplying out terms:

\[
A \left( \frac{q^2 - p^2}{p \exp(-ik_{111}a) + q \exp(ik_{111}a)} \right) = \frac{C}{k_{111} [p \exp(-ik_{111}a) + q \exp(ik_{111}a)]} \exp(ik_Ia)
\]

\[
\left( \frac{C}{A} \right) = \frac{k_{111} \exp(-ik_Ia) (q^2 - p^2)}{p \exp(-ik_{111}a) \left[ 1 - \frac{k_I}{k_{111}} \right] + q \exp(ik_{111}a) \left[ 1 + \frac{k_I}{k_{111}} \right]}
\]

\[
\left( \frac{C}{A} \right) = \frac{\exp(-ik_Ia) (q^2 - p^2)}{-p^2 \exp(-ik_{111}a) + q^2 \exp(ik_{111}a)}
\]

With $q^2 - p^2 = -4k_I/k_{111}$, we have:

\[
\left( \frac{C}{A} \right) = \frac{4 \frac{k_I}{k_{111}} \exp(-ik_Ia)}{p^2 \exp(-ik_{111}a) - q^2 \exp(ik_{111}a)} \tag{17}
\]
Now that we have an expression for \( C/A \), we need to substitute Equation 17 into the transmission coefficient equation and replace \( p \) and \( q \) with their respective values:

\[
T = \frac{v_I C^* C}{v_I A^* A} = \left( \frac{4 k_{II} \exp (ik_I a)}{p^2 \exp (ik_{III} a) - q^2 \exp (-ik_{III} a)} \right) \left( \frac{4 k_{II} \exp (-ik_I a)}{p^2 \exp (-ik_{III} a) - q^2 \exp (ik_{III} a)} \right)
\]

\[
= \frac{16 \left( \frac{k_I}{k_{III}} \right)^2}{(p^2 - q^2)^2 - p^2 q^2 [\exp (ik_{III} a) + \exp (-ik_{III} a)]^2}
\]

Utilizing the \( p/q \) relationships in the denominator once more, we have:

\[
T = \frac{16 \left( \frac{k_I}{k_{III}} \right)^2}{16 \left( \frac{k_I}{k_{III}} \right)^2 - \left( 1 - \frac{k_I^2}{k_{III}^2} \right) [\exp (ik_{III} a) + \exp (-ik_{III} a)]^2}
\]

We can now use the relationship:

\[
\left( \frac{k_I}{k_{III}} \right)^2 = \frac{E}{E - V_o}
\]

to obtain a slightly modified version of Equation 6-51:

\[
T = \left[ 1 - \frac{[\exp (ik_{III} a) + \exp (-ik_{III} a)]^2}{16 \frac{E}{V_o} \left( \frac{E}{V_o} - 1 \right)} \right]
\]

**Part B** By substituting \( k_{II} \) with \( ik_{III} \) in Equation 6-49:

\[
T = \left[ 1 - \frac{[\exp (ik_{III} a) + \exp (-ik_{III} a)]^2}{16 \frac{E}{V_o} \left( \frac{E}{V_o} - 1 \right)} \right]
\]

\[
\Rightarrow \left[ 1 - \frac{[\exp (ik_{III} a) + \exp (-ik_{III} a)]^2}{16 \frac{E}{V_o} \left( \frac{E}{V_o} - 1 \right)} \right]
\]

we have the same expression that was found in Part A.

---

1See E&R page 201 for this expression.
The transmission coefficients from 6-49 and 6-50 are:

\[
T = \left( 1 + \frac{\sinh^2 (k_{II}a)}{4E Vo \left( 1 - \frac{E}{V_o} \right)} \right)^{-1}
\]
(6-49)

\[
T \approx 16 \frac{E}{V_o} \left( 1 - \frac{E}{V_o} \right) \exp (-2k_{II}a)
\]
(6-50)

**Part A** For a 2 eV electron incident on a rectangular potential barrier of height 4 eV and thickness \(10^{-10}\) m, \(k_{II}a\) is:

\[
k_{II}a = \sqrt{\frac{2mV_o a^2}{\hbar^2} \left( 1 - \frac{E}{V_o} \right)}
\]

\[
= \sqrt{\frac{2 \cdot 9.1 \times 10^{-31} \cdot 4 \cdot (10^{-10})^2 \cdot 1.602 \times 10^{-19}}{(1.055 \times 10^{-34})^2} \left( 1 - \frac{2}{4} \right)}
\]

\[
= 0.724
\]

With this quantity, the transmission coefficient from 6-49 equals:

\[
T = \left( 1 + \frac{\sinh^2 (0.724)}{4^2 \left( 1 - \frac{2}{4} \right)} \right)^{-1}
\]

\[
= 0.6164
\]

and 6-50 equals:

\[
T \approx 16 \cdot \frac{2}{4} \left( 1 - \frac{2}{4} \right) \exp (-2 \cdot 0.724)
\]

\[
= 0.9402
\]

The approximation for the transmission coefficient is not appropriate in this case since \(T\) is not very small.

**Part B**

\[
k_{II}a = \sqrt{\frac{2 \cdot 9.1 \times 10^{-31} \cdot 4 \cdot (9 \times 10^{-9})^2 \cdot 1.602 \times 10^{-19}}{(1.055 \times 10^{-34})^2} \left( 1 - \frac{2}{4} \right)}
\]

\[
= 65.144
\]

\[
T = \left( 1 + \frac{\sinh^2 (65.144)}{4^2 \left( 1 - \frac{2}{4} \right)} \right)^{-1}
\]

\[
= 1.044 \times 10^{-56}
\]

\[
T \approx 16 \cdot \frac{2}{4} \left( 1 - \frac{2}{4} \right) \exp (-2 \cdot 65.144)
\]

\[
= 1.044 \times 10^{-56}
\]
Part C

\[ k_{11a} = \sqrt{\frac{2 \cdot 9.1 \times 10^{-31} \cdot 4 \cdot (10^{-9})^2 \cdot 1.602 \times 10^{-19}}{(1.055 \times 10^{-34})^2} \left( 1 - \frac{2}{4} \right)} \]

\[ = 7.238 \]

\[ T = \left(1 + \frac{\sinh^2 (7.238)}{\frac{4}{3} (1 - \frac{2}{4})}\right)^{-1} \]

\[ = 2.066 \times 10^{-6} \]

\[ T \approx 16 \cdot \frac{2}{4} \left( 1 - \frac{2}{4} \right) \exp (-2 \cdot 7.238) \]

\[ = 2.066 \times 10^{-6} \]

---

E&R - Ch. 6, Problem 19

The standing wave general solution is:

\[ \psi(x) = A \sin (kx) + B \cos (kx) \]

To verify that this is a valid solution, we need to substitute it into the time-independent Schrödinger equation (TISE) and determine if an equality exists. Looking at the second spatial derivative, we have:

\[ \frac{d^2 \psi(x)}{dx^2} = -Ak^2 \sin (kx) - Bk^2 \cos (kx) \]

The TISE for the standing wave general solution is then:

\[ -\frac{\hbar^2}{2m} \left( -Ak^2 \sin (kx) - Bk^2 \cos (kx) \right) = E\psi(x) \]

\[ \frac{\hbar^2 k^2}{2m} \left( A \sin (kx) + B \cos (kx) \right) = E\psi(x) \]

\[ \frac{\hbar^2 k^2}{2m} \psi(x) = E\psi(x) \]

Therefore, the standing wave general solution is a solution to the Schrödinger equation due to the existence of valid eigenvalues (or energy values):

\[ E = \frac{\hbar^2 k^2}{2m} \]

■
Problem 5

In this problem, we consider an infinite spherical potential well with the following properties:

\[ V(r) = \begin{cases} 0 & \text{for } 0 < r < r_o \\ \infty & \text{for } r > r_o \end{cases} \]

In the lowest energy state, we know that \( n = 1 \) and \( l, m_l = 0 \). Inside the spherical well, we also have \( V = 0 \). With these values, the spherical form of the time-independent Schrödinger wave equation can be simplified:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2\mu r^2}{\hbar^2} E = -\frac{m_l^2}{\sin^2(\theta)} - \frac{1}{\Theta \sin(\theta)} \frac{d}{d\theta} \left( \sin(\theta) \frac{d\Theta}{d\theta} \right)
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2\mu R}{\hbar^2} E = l(l+1) \frac{R}{r^2}
\]

\[ \Rightarrow \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2\mu R}{\hbar^2} E = 0 \]

We can utilize the definitions provided in the problem, namely that the radial wave function \( R(r) = \chi(r)/r \). Evaluating the derivative on the right hand side, we have:

\[
\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \left( \frac{\chi(r)}{r} \right) \right] + \frac{2\mu \chi(r)}{\hbar^2 r} E = 0
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \left( \frac{1}{r} \frac{d\chi(r)}{dr} - \frac{\chi(r)}{r^2} \right) \right] + \frac{2\mu \chi(r)}{\hbar^2 r} E = 0
\]

\[
\frac{1}{r^2} \left( \frac{d^2 \chi(r)}{dr^2} r + \frac{d\chi(r)}{dr} - \frac{d\chi(r)}{dr} \right) + \frac{2\mu \chi(r)}{\hbar^2 r} E = 0
\]

\[ \Rightarrow \frac{d^2 \chi(r)}{dr^2} + \frac{2\mu}{\hbar^2} E \chi(r) = 0 \]

The final equation is nearly identical in form to the rectangular Schrödinger equation in free space. Utilizing the boundary conditions, \( \chi(r = r_o) = 0 \) and \( \chi(0) = 0 \), we find that the problem is similar to that of a infinite 1-D potential well which has the well known eigenenergy value:

\[ E_n = \frac{\hbar^2 k_n^2}{2\mu} \]

where \( k_n = \frac{n\pi}{r_o} \) for the infinite spherical well. For the case of \( n = 1 \),

\[ E = \frac{\hbar^2 \pi^2}{2\mu r_o^2} \text{ for the lowest energy state.} \]
Problem 6

The wave equation for the ground state of the hydrogen atom \((Z = 1)\) and potential is:

\[
\psi_{100} = \frac{1}{\sqrt{\pi}} \left( \frac{1}{a_o} \right)^{3/2} \exp \left( -\frac{r}{a_o} \right)
\]

\[
V(r) = -\frac{q^2}{4\pi\epsilon_0 r}
\]

where \(a_o\) is the Bohr radius and equal to:

\[
\frac{4\pi\epsilon_0 h^2}{\mu q^2}
\]

**Part A**  The expectation value for potential, \(\langle V \rangle\), is:

\[
\langle V \rangle = \int_0^\infty \int_0^{\pi} \int_0^{2\pi} \psi_{100}^* (r) V(r) \psi_{100} (r) r^2 \sin(\theta) \, dr \, d\theta \, d\phi
\]

If we take out constants, the integral can be simplified to:

\[
\langle V \rangle = -\frac{q^2}{4\pi^2\epsilon_0 a_o^3} \int_0^\infty \int_0^{\pi} \int_0^{2\pi} \exp \left( -\frac{r}{a_o} \right) \cdot \frac{1}{r} \cdot \exp \left( -\frac{r}{a_o} \right) r^2 \sin(\theta) \, dr \, d\theta \, d\phi
\]

Evaluating the \(d\theta\) and \(d\phi\) parts of the integral, we have:

\[
\langle V \rangle = -\frac{q^2}{4\pi^2\epsilon_0 a_o^3} \int_0^\infty \int_0^{\pi} \int_0^{2\pi} r \exp \left( -\frac{2r}{a_o} \right) \sin(\theta) \, dr \, d\theta \, d\phi
\]

The last simplification is to do a substitution with \(u = 2r / a_o\), so:

\[
u = \frac{2r}{a_o}
\]

\[
du = \frac{2}{a_o} \, dr
\]

\[
\leftrightarrow \, dr = \frac{a_o}{2} \, du
\]

The integral then becomes:

\[
\langle V \rangle = -\frac{q^2}{4\pi^2\epsilon_0 a_o^3} \int_0^\infty \frac{a_o u}{2} \exp (-u) \, a_o \, du
\]

\[
= -\frac{q^2}{4\pi^2\epsilon_0 a_o} \int_0^\infty u \exp (-u) \, du
\]

\[
= -\frac{q^2}{4\pi^2\epsilon_0 a_o}
\]

since \(\int_0^\infty u \exp (-u) \, du = 1\). Substituting in the Bohr radius, we obtain:

\[
\langle V \rangle = -\frac{\mu q^4}{(4\pi\epsilon_0)^2 h^2}
\]
**Part B**  From Equation 7-22, the energy of the hydrogen atom is:

\[ E_n = -\frac{\mu q^4}{(4\pi\varepsilon_0)^2 2\hbar^2 n^2} \]

For \( n = 1 \), we see that \( E_1 = \langle V \rangle / 2 \).

**Part C**  For total energy, we have \( E = K + V \). Using the expectation value from before, total energy can be expressed as a sum of the expectation values for potential and kinetic energy:

\[ E = \langle K \rangle + \langle V \rangle \]

But,

\[ E = \frac{\langle V \rangle}{2} \]

So we have:

\[ \frac{\langle V \rangle}{2} = \langle K \rangle + \langle V \rangle \]

Therefore,

\[ \langle K \rangle = -\frac{\langle V \rangle}{2} \]