The Wave Equation in Birefringent Media, Modes in Optical Fiber

Read: Kasap, Chapter 2
Homework#1 due Wednesday

ECE 162C

Lecture #3

Prof. John Bowers

Maxwell's Equations

$$\nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t}$$

$$\nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t}$$

$$\nabla \bullet \vec{d} = 0$$

$$\nabla \bullet \vec{b} = 0$$

where e and h are the electric and magnetic field vectors
d and b are the electric and magnetic displacement vectors
No free charge.

Constitutive Relations

$$\vec{d} = \varepsilon_0 \vec{e} + \vec{p}$$

$$\vec{b} = \mu_0 (\vec{h} + \vec{m})$$

p and m are the electric and magnetic polarizations of the medium
ε_0 and μ_0 are the electric and magnetic permeabilities of vacuum
e and h are the electric and magnetic field vectors
d and b are the electric and magnetic displacement vectors

Electric Susceptibility χ (Anisotropic media)

Anisotropic Media: χ is a complex second rank tensor

$$\vec{P} = \varepsilon_0 \overleftrightarrow{\chi} \vec{E}$$

$$P_i = \varepsilon_0 \sum \chi_{ij} E_j$$

$$P_x = \varepsilon_0 (\chi_{xx} E_x + \chi_{xy} E_y + \chi_{xz} E_z)$$

One can always choose a coordinate system such that off axis elements are zero. These are the principal dielectric axes of the crystal. We will only use the principal coordinate system.

$$P_x = \varepsilon_0 \chi_{11} E_x$$

$$P_y = \varepsilon_0 \chi_{22} E_y$$

$$P_z = \varepsilon_0 \chi_{33} E_z$$

Principal Axes

D, E and P are not parallel in general. D and E are related by the electric permeability tensor ε

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{D} = \varepsilon \vec{E}$$

Principal axes can always be chosen such that D and E are parallel and the off diagonal elements of ε are zero.

$$\varepsilon_{11} = \varepsilon_0 (1 + \chi_{11})$$

$$\varepsilon_{22} = \varepsilon_0 (1 + \chi_{22})$$

$$\varepsilon_{33} = \varepsilon_0 (1 + \chi_{33})$$

Wave Propagation in Lossless, Isotropic Media

- Lossless: σ=0, χ is real, ε is real.
- Isotropic: χ, ε are scalors (not tensors).

$$\nabla \times \vec{e} = i + \frac{\partial \vec{b}}{\partial t} = 0 + \mu \frac{\partial \vec{h}}{\partial t}$$

$$\nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{e}) = \mu \frac{\partial (\nabla \times \vec{h})}{\partial t} = \mu \frac{\partial^2 \vec{d}}{\partial^2 t} = \mu\varepsilon \frac{\partial^2 \vec{e}}{\partial^2 t}$$

$$\nabla \times (\nabla \times \vec{e}) = \nabla^2 \vec{e} - \nabla(\nabla \bullet \vec{e})$$

$$\nabla^2 \vec{e} = \mu\varepsilon \frac{\partial^2 \vec{e}}{\partial^2 t}$$

Wave Equation

Wave Equation

$$e(x, y, z, t) = \mathrm{Re}[E(x, y, z)e^{i\omega t}]$$

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0$$

$$\boxed{\nabla^2 \vec{E} + k^2 \vec{E} = 0}$$

where

$$k = \omega\sqrt{\mu\varepsilon} = \omega n / c$$

$$c = 1/\sqrt{\mu_0 \varepsilon_0}$$

$$n = \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$$

Step Index Circular Waveguide (lossless, isotropic)

- Simplest type of fiber
- (Most fiber these days is far more complex)
- Cylindrical symmetry
- Express Laplacian operator in cylindrical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

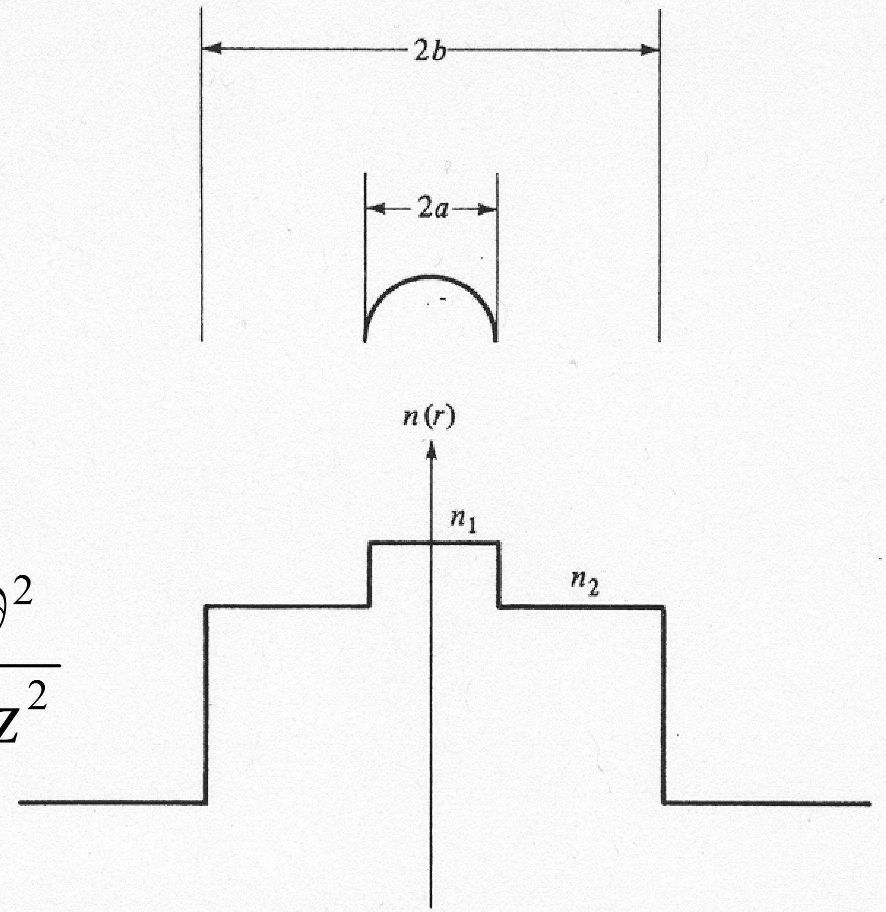


Figure 3-1 Structure and index profile of a step-index circular wa

- Separate variables

$$E_r = \psi(r)\Phi(\phi)e^{i(\omega t - \beta z)}$$

Separable Solutions

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + (k^2 - \beta^2)]E_z = 0$$

$$E_r = \psi(r)\Phi(\phi)e^{i(\omega t - \beta z)}$$

$$\Phi(\phi) = e^{\pm il\phi} \quad \text{where} \quad l = 0,1,2...$$

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + (k^2 - \beta^2 - \frac{l^2}{r^2})]\psi = 0$$

Bessel differential equation

$$\psi = c_1 J_l(hr) + c_2 Y_l(hr) \quad k^2 - \beta^2 = h^2 > 0$$

$$\psi = c_1 I_l(qr) + c_2 K_l(qr) \quad k^2 - \beta^2 = -q^2 > 0$$

J Bessel function of the first kind

Y Bessel function of the second kind

I Modified Bessel function of the first kind

K Modified Bessel function of the second kind

Separable Solutions

Bessel differential equation

$$\psi = c_1 J_l(hr) + c_2 Y_l(hr) \quad k^2 - \beta^2 = h^2 > 0$$

$$\psi = c_1 I_l(qr) + c_2 K_l(qr) \quad k^2 - \beta^2 = -q^2 > 0$$

J Bessel function of the first kind
Y Bessel function of the second kind
I Modified Bessel function of the first kind
K Modified Bessel function of the second kind

Boundary Conditions

Decaying fields for r>a
q>0

$$q^2 = \beta^2 - k^2 = \beta^2 - n_2^{\ 2} k_0^{\ 2}$$
$$k_0 = \omega / c$$

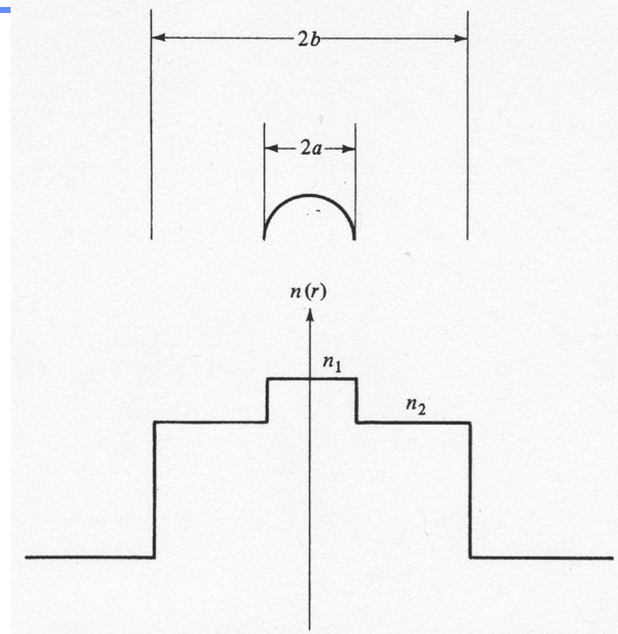


Figure 3-1 Structure and index profile of a step-index circular waveguide.

For fields in the core r<a, we need finite fields
(which eliminates Y and K which go to infinity as r approaches 0.

Fiber Modes

- Mode: A particular electromagnetic field configuration. For a given electromagnetic problem, many field distributions exist that satisfy the wave equation, Maxwell's equations, and the boundary conditions.
- The particular distribution does not change with propagation. (Only phase changes).
- Each β_{mn} corresponds to a mode.

Mode Names

Name	Field Components	Notation
transverse-electric	E_z=0	TE_{0n}
transverse-magnetic	H_z=0	TM_{0n}
hybrid	H_z dominates E_z dominates	HE_{mn} EH_{mn}
linearly polarized	E_z and H_z close to zero	LP_{mn}

TE (m=0) Modes

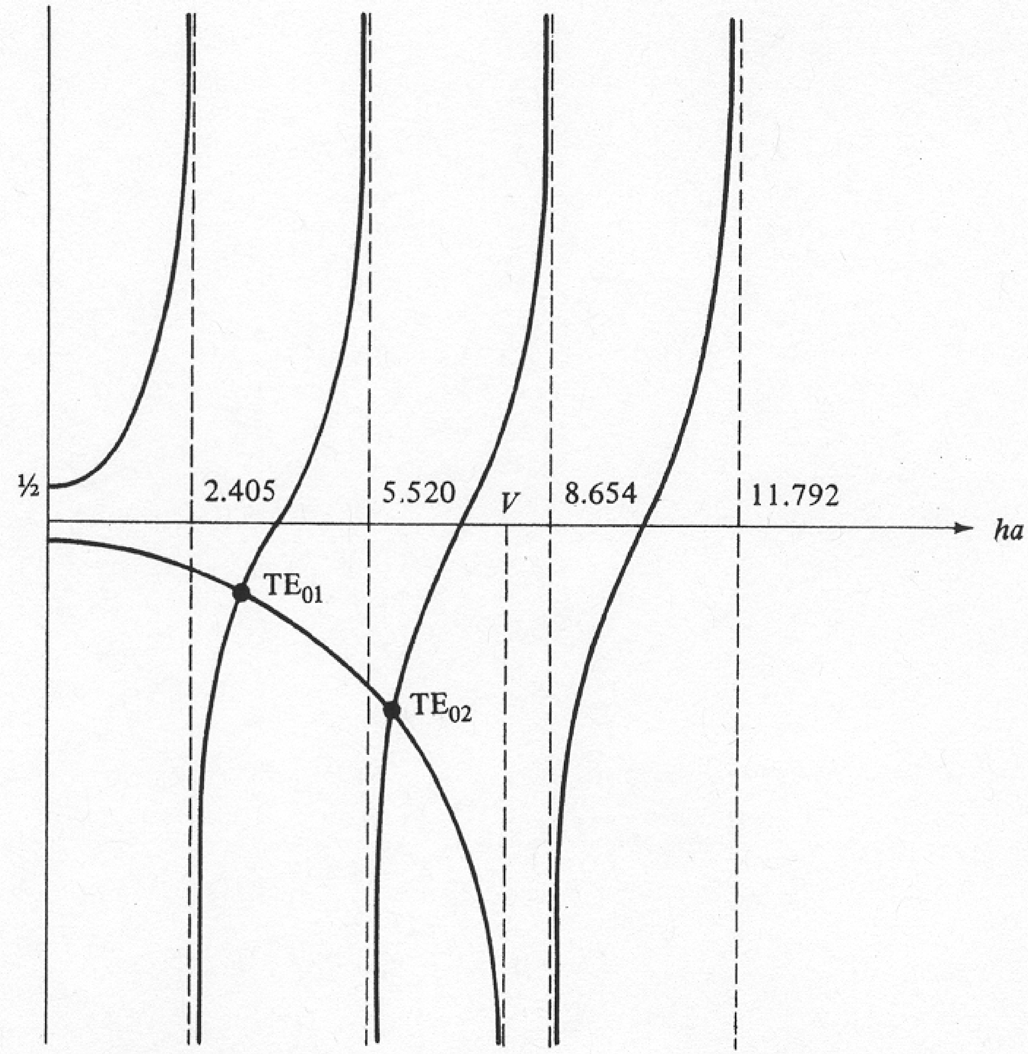


Figure 3-2 Graphical determination of the propagation constants of TE modes ($l = 0$) for a step-index waveguide.

EH (m=1)

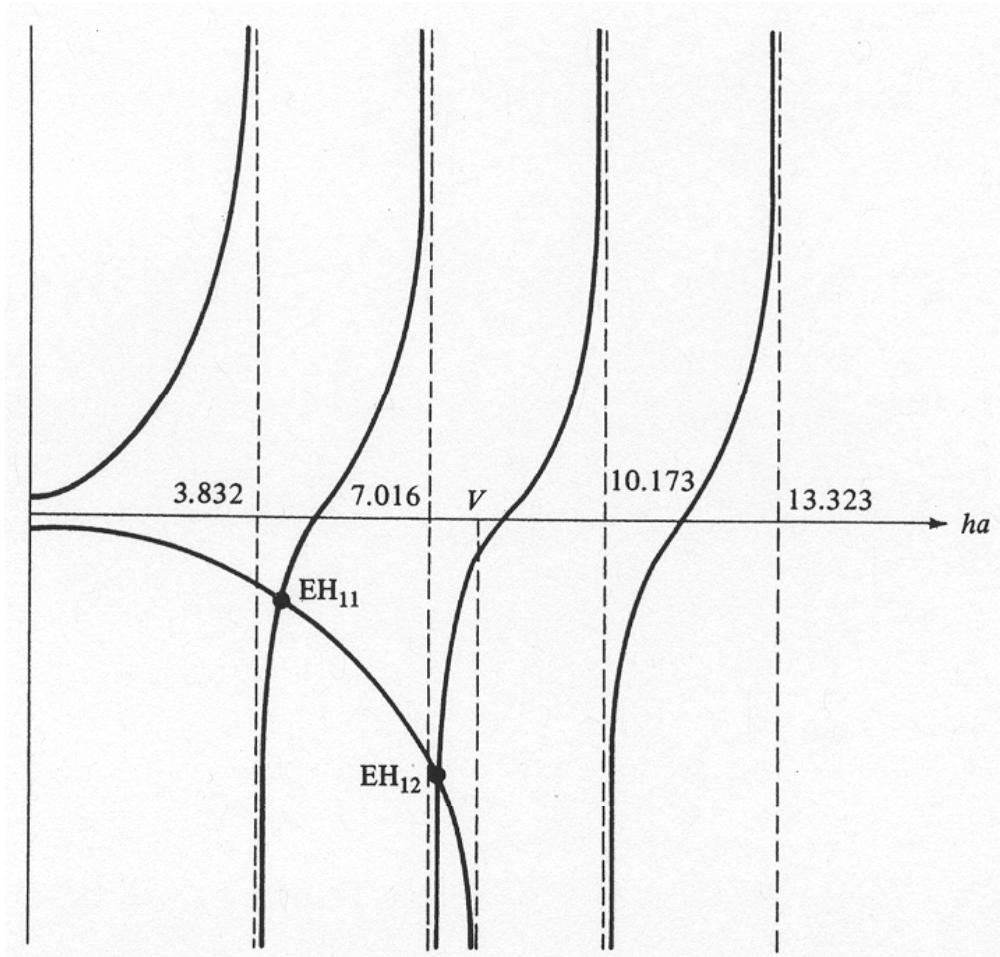


Figure 3-3 Graphical determination of the propagation constants of $l = 1$ EH modes for a step-index fiber.

HE (m=1)

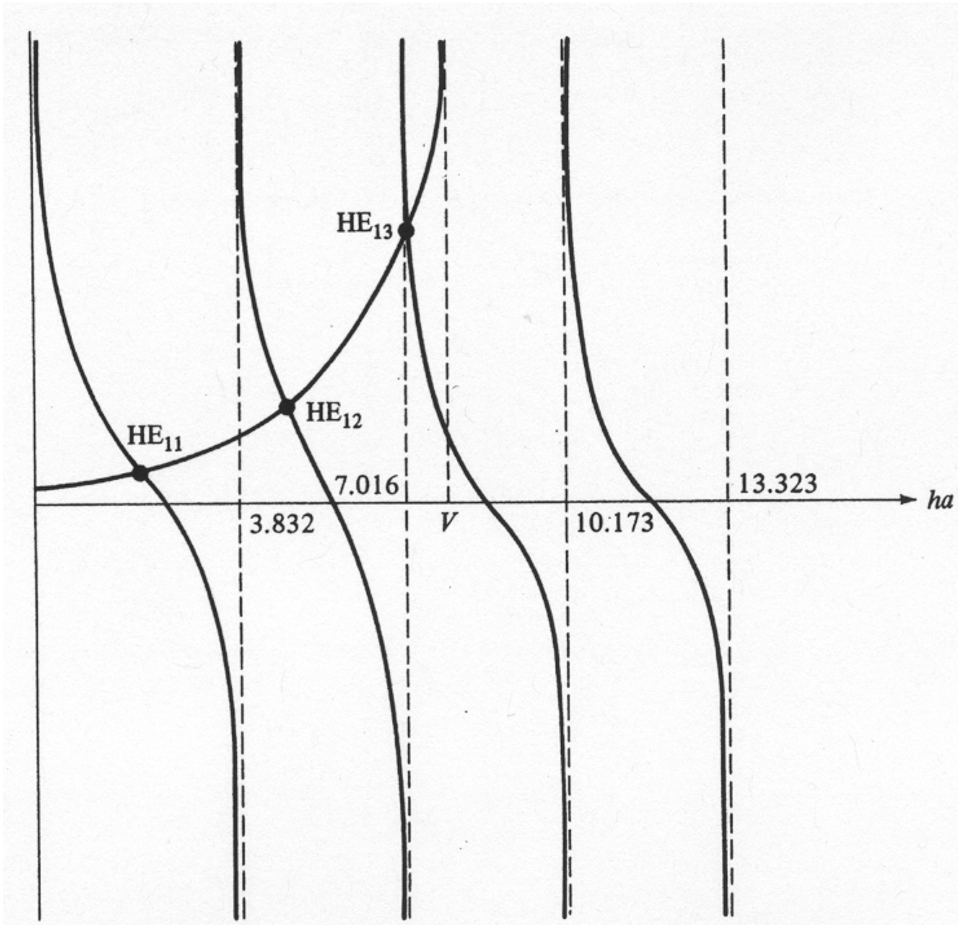


Figure 3-4 Graphical determination of the propagation constants of the $l = 1$ HE modes for a step-index dielectric waveguide.

V parameter

$\overline{n}$: effective index

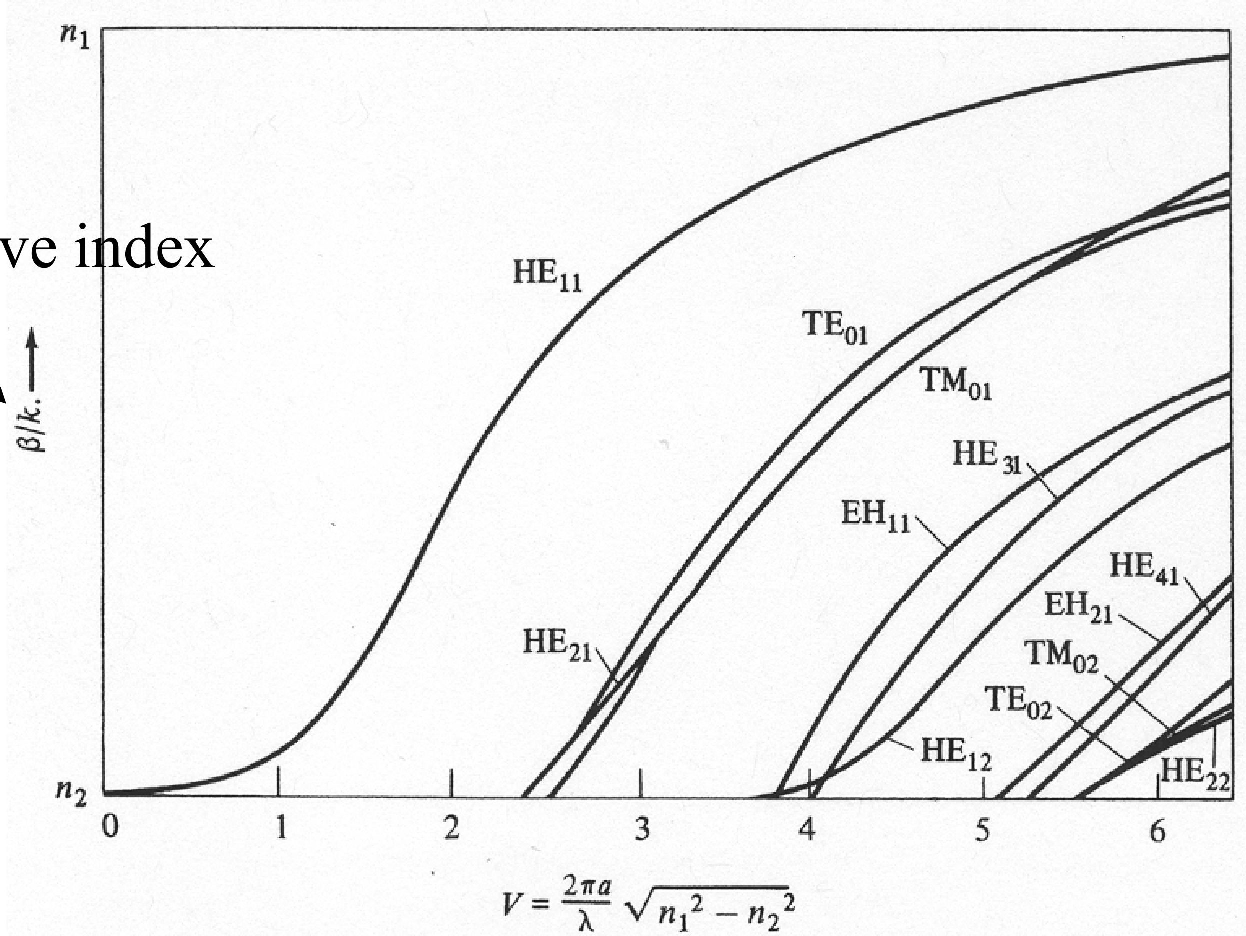


Figure 3-5 Normalized propagation constant as a function of V parameter for a few of the lowest-order modes of a step-index waveguide [4].

For n1-n2<<n1, LP approximation is valid.

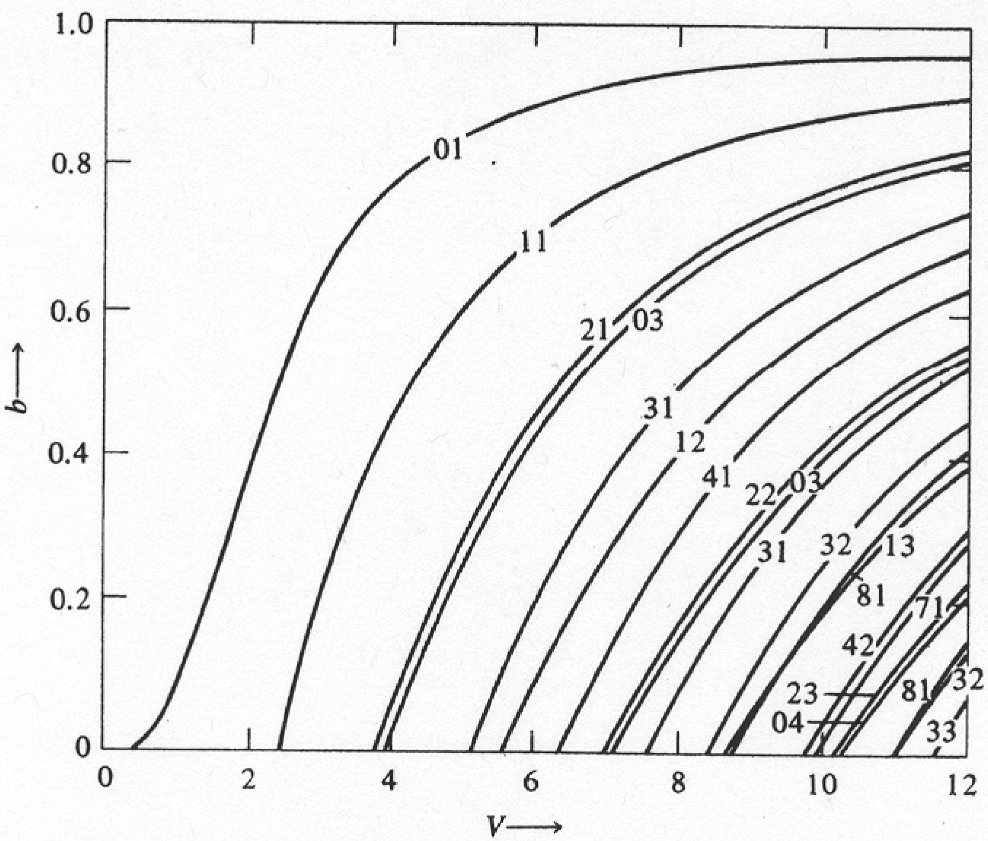


Figure 3-6 Normalized propagation constant b as function of normalized frequency V for the guided modes of the optical fiber, $b = (\beta/k_c - n_2)/(n_1 - n_2)$. (After Reference [5].)

Single mode cut off: V=2.405

Modes as a function of V parameter

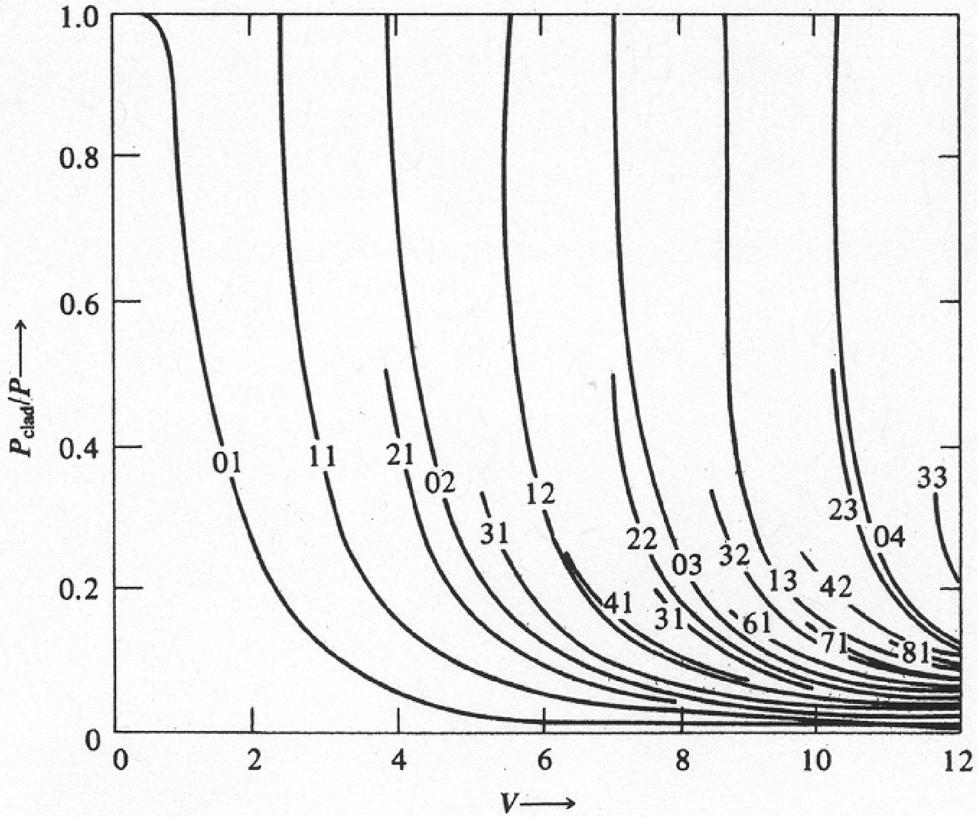


Figure 3-9 Fractional power contained in the cladding as a function of the frequency parameter *V*. (After Reference [5].)

At cutoff, all the power is in the cladding.

Degenerate Modes LP_{11}

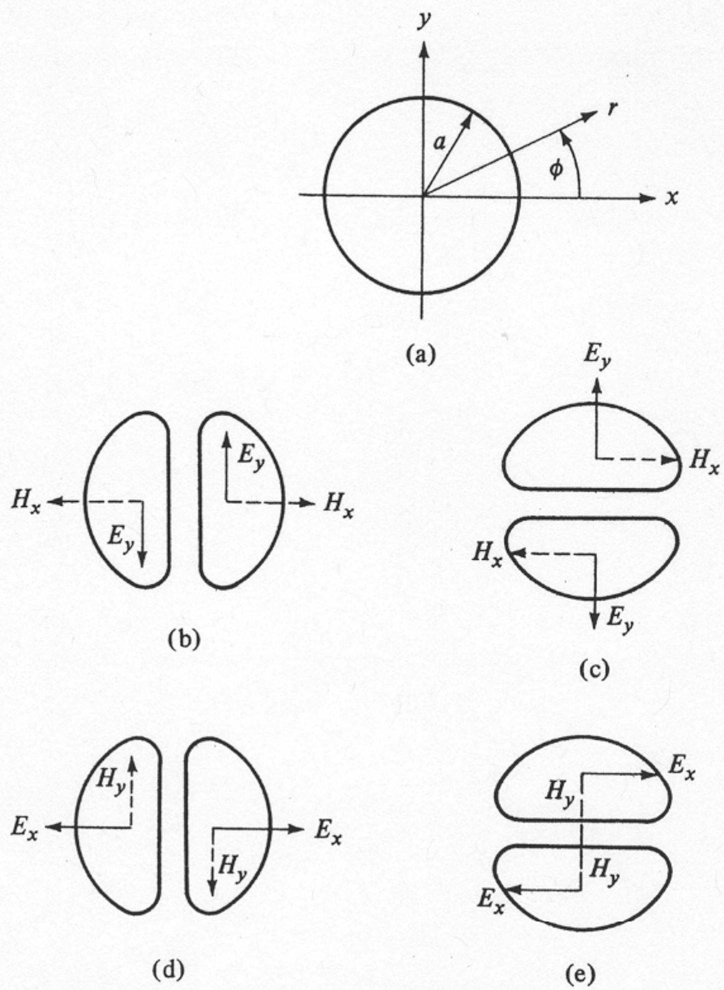


Figure 3-8 Sketch of the fiber cross section and the four possible distributions of LP_{11}.

Common fiber types

- Glass fiber:
  - Step index: n_1=1.48 n_2=1.46
    - Ge doped core
    - Depressed cladding (Fluorine doped)
    - Amplifier (Er doped)
  - Graded index fiber: profile designed so all modes travel at the same velocity
  - Polarization preserving fiber (strain or ellipticity to break the degeneracy between the two polarization modes.
  - Dispersion management:
    - Dispersion shifted fiber
    - Dispersion flattened fiber
- Plastic fiber (low cost, automotive, stereo, etc.) Polymethyl methacrylate core

Fiber-Optic Waveguides

- Step index fiber: Standard for single mode (small core size – 8 micron)
- Graded index fiber: Designed so all multimodes travel at the same velocity.

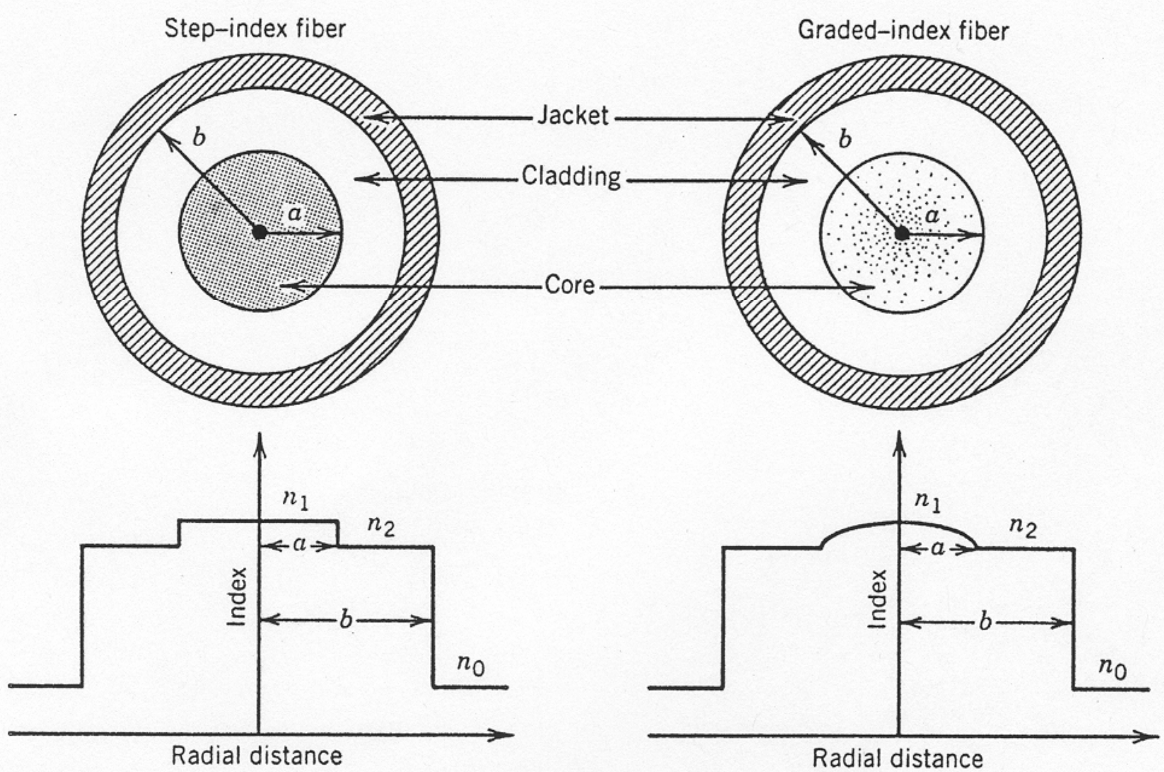


Figure 2.1: Cross section and refractive-index profile for step-index and graded-index fibers.

Two primary limits to transmission

- Loss: Loss budget for loss limited transmission
- Dispersion: Dispersion budget for dispersion limited transmission.

Comparison to cable

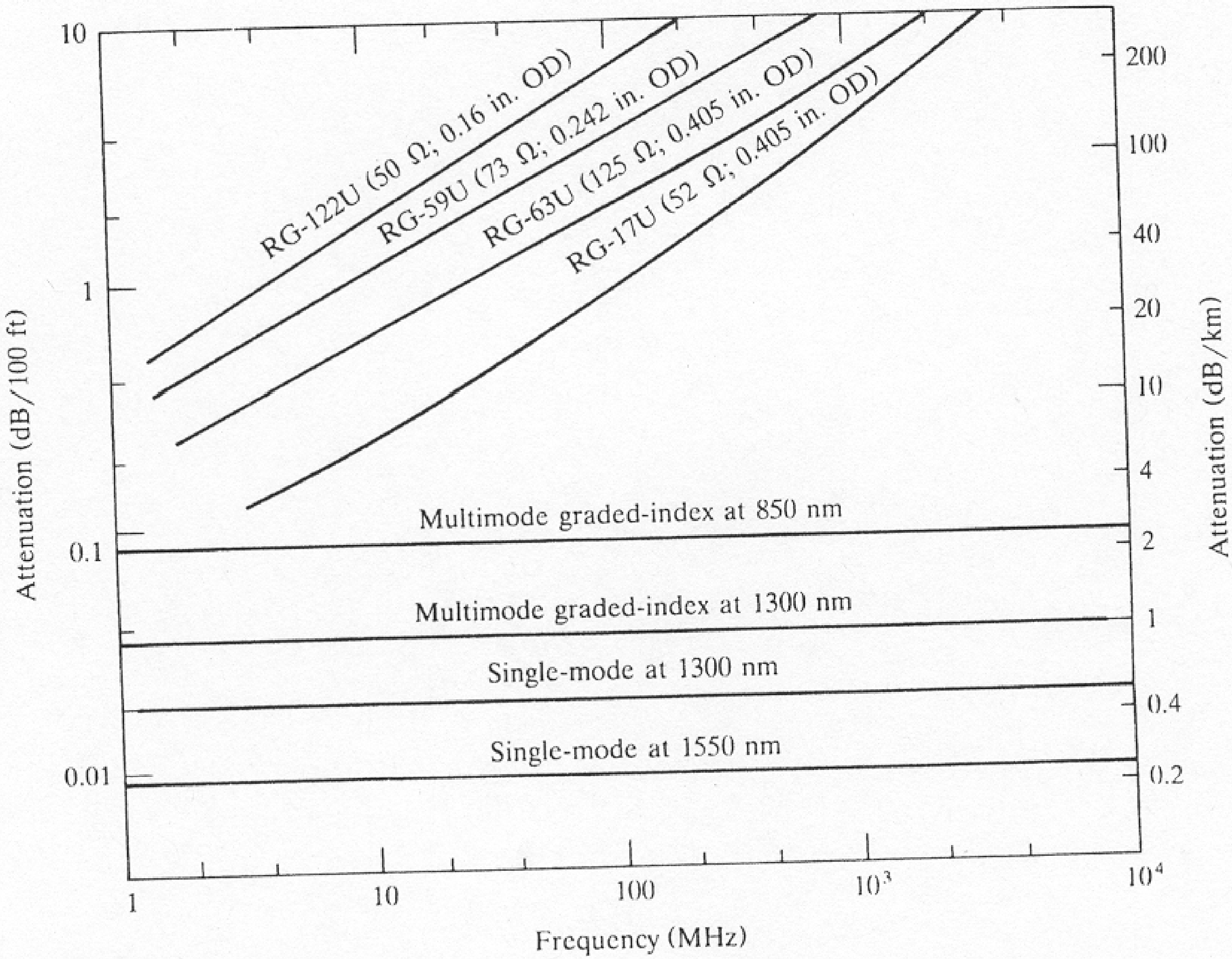


FIGURE 3-11
A comparison of the attenuation as a function of frequency or data rate of various coaxial cables and several types of high-bandwidth optical fibers.

E

Loss in early optical fibers
(now the O-H peaks around 1.4 μm are small)

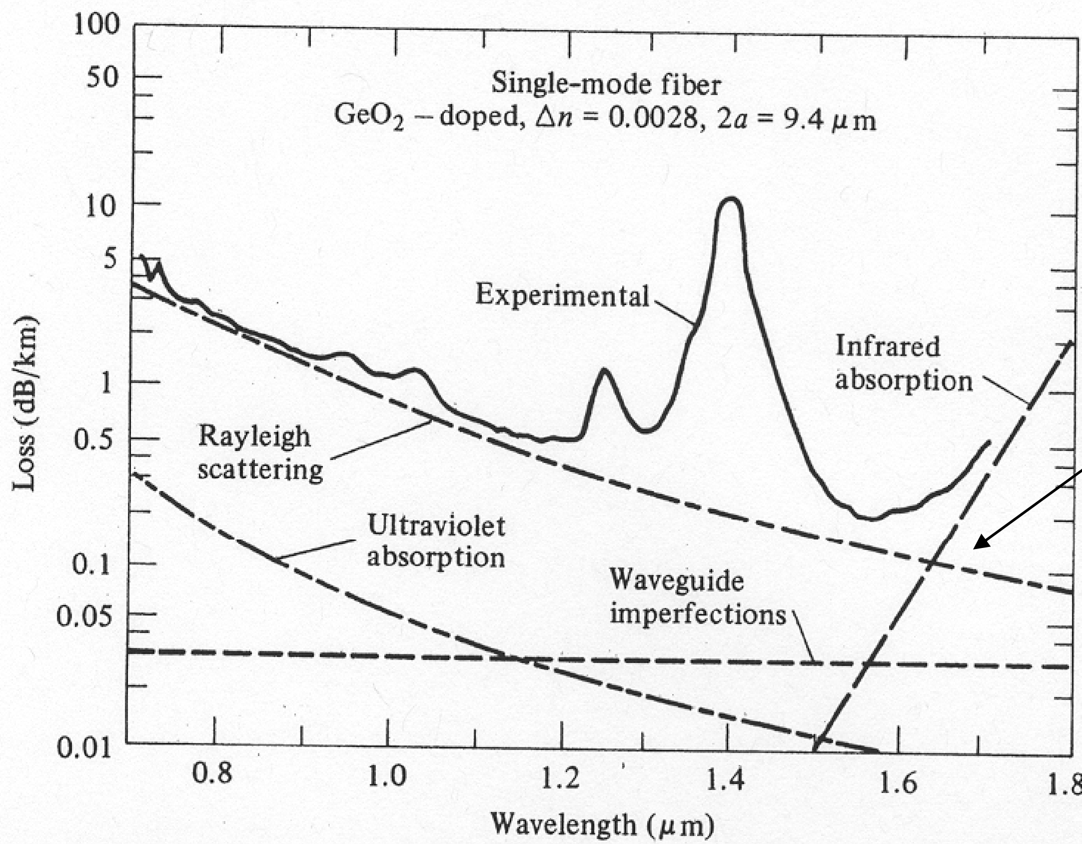


Figure 3-19 Observed loss spectrum of a germanosilicate single-mode fiber. Estimated loss spectra for various intrinsic materials effects and waveguide imperfections are also shown. (From Reference [20].)

$$\alpha_R = C / \lambda^4$$

Two primary limits to transmission

- Loss: Loss budget for loss limited transmission
- Dispersion: Dispersion budget for dispersion limited transmission.

Loss Budget

- p_{trans}=transmitter power
- p_{rec}=sensitivity of receiver

$$p_{rec} = p_{trans} e^{-\alpha L}$$

- Take 10 log of each side and express in dBm
- P_{trans}, P_{rec}

$$P_{rec} = P_{trans} - \alpha L$$

$$L_{\max} = \frac{P_{trans} - P_{rec}}{\alpha}$$

- Example:
- P_{trans}=10 dBm
- P_{rec}=-20 dBm
- L_{max}=30 dB/0.2 dB/km=150 km

Dispersion

- Multimode– different modes have different β
- Intramodal (i.e. group-velocity dispersion)
  - Material dispersion – silica refractive index is a function of wavelength
  - Waveguide dispersion – V parameter is a function of wavelength
- Polarization-Mode Dispersion – bifrefringence induced by perturbations

Multimode Dispersion

- For step index multimode fibers, the fiber bandwidth (in MHz km) is given by

$$B < \frac{n_2}{n_1^2}\frac{c}{L\Delta}$$

- For graded index fibers, the fiber bandwidth in MHz km is given by

$$B < \frac{8c}{n_1 L\Delta^2}$$

Fiber-Optic Waveguides

- Step index fiber: Standard for single mode (small core size – 8 micron)
- Graded index fiber: Designed so all multimodes travel at the same velocity.

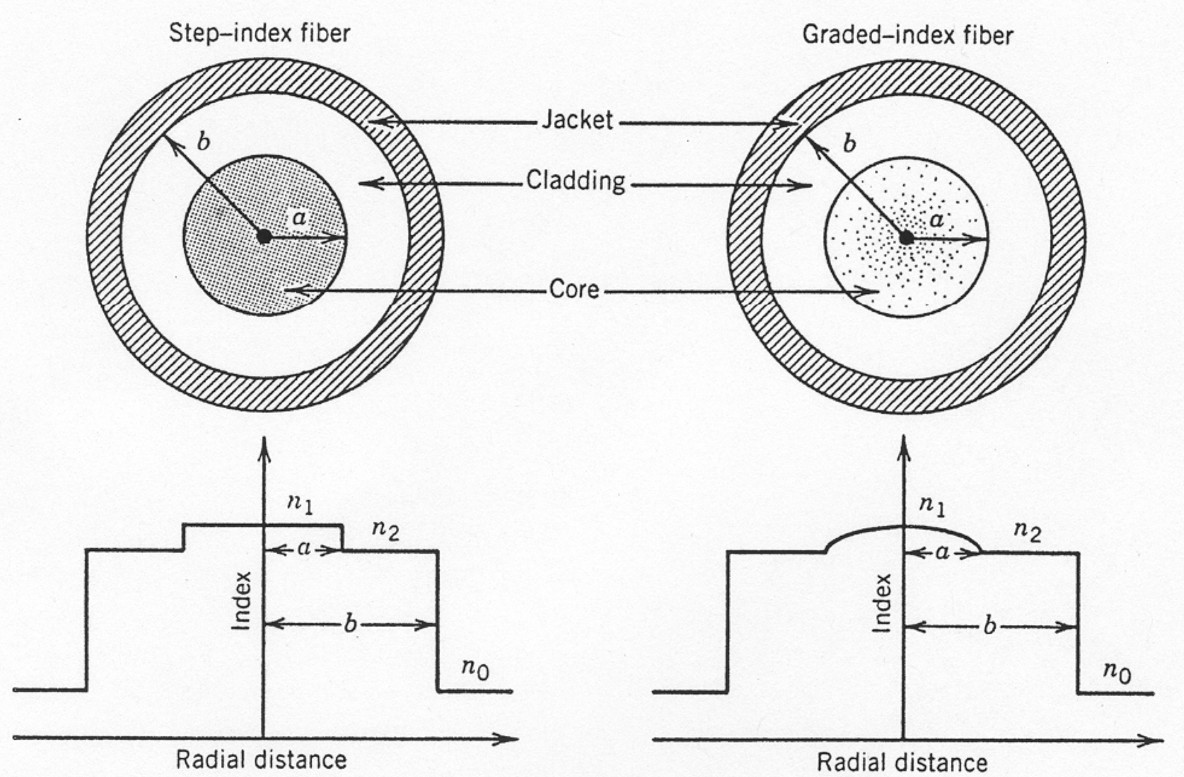


Figure 2.1: Cross section and refractive-index profile for step-index and graded-index fibers.

Group-Velocity Dispersion

- The index of the mode is dependent on the wavelength (i.e. the fiber is dispersive).
- Two components: material dispersion and waveguide dispersion.
- These contribute to phase index.
- The group index is given by

$$n_g = n + \omega \frac{\partial n}{\partial \omega}$$

$$D = -\frac{2\pi c}{\lambda^2}\frac{d^2\beta}{d\omega^2} = -\frac{2\pi c}{\lambda^2}\beta_2$$

Units are ps/(km-

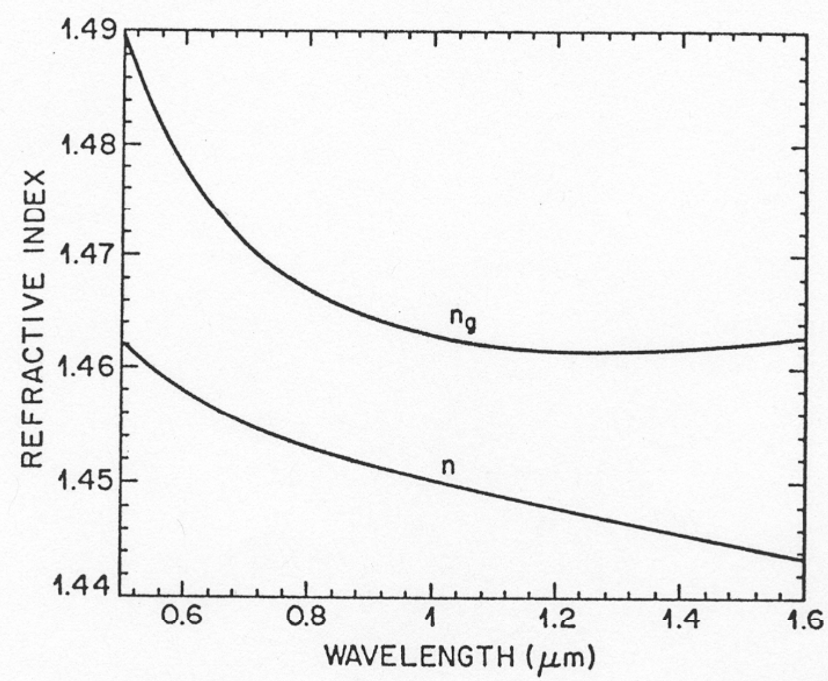


Figure 2.8: Variation of refractive index n and group index n_g with wavelength for fused silica.

Material Dispersion

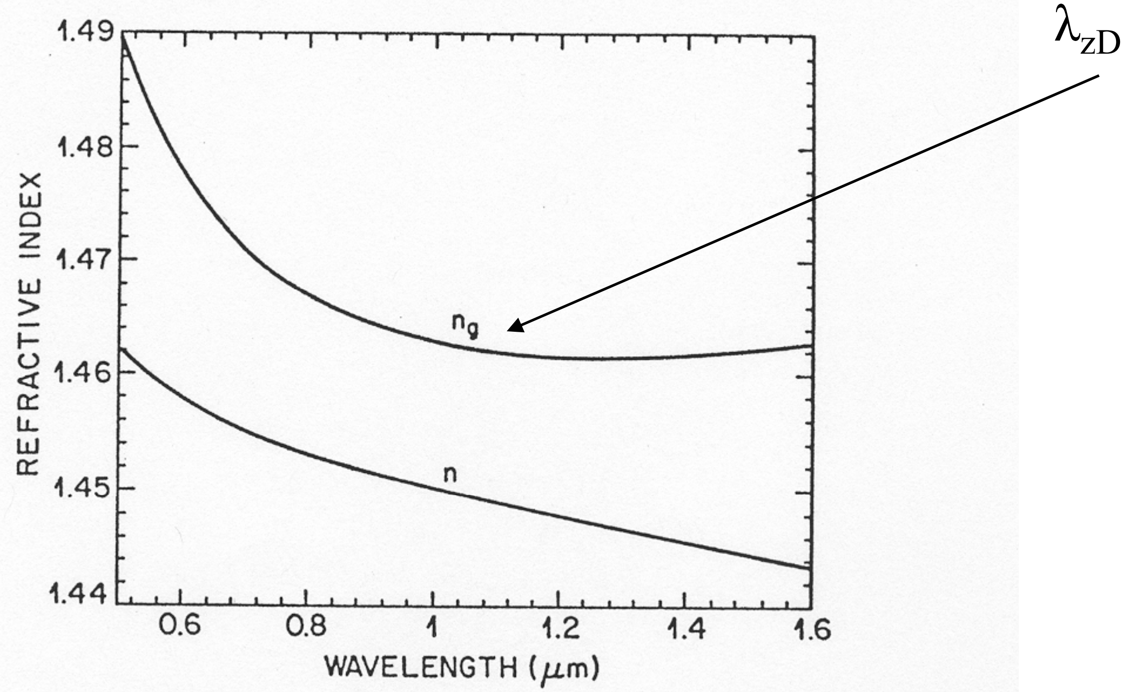
- Refractive index change of silica with optical frequency is modeled with the Sellmeier Equation:

$$n^2(\omega) = 1 + \sum_{j=1}^{M} \frac{B_j \omega_j}{\omega_j^2 - \omega^2}$$

B_j is the strength of medium resonance j of the material
ω_j is the frequency of medium resonance j

Material Dispersion

- Material dispersion D_M is the slope of the n_g vs. λ (times 1/c)
- Therefore, looking at the figure we see that the slope hits zero at some wavelength – zero-dispersion wavelength

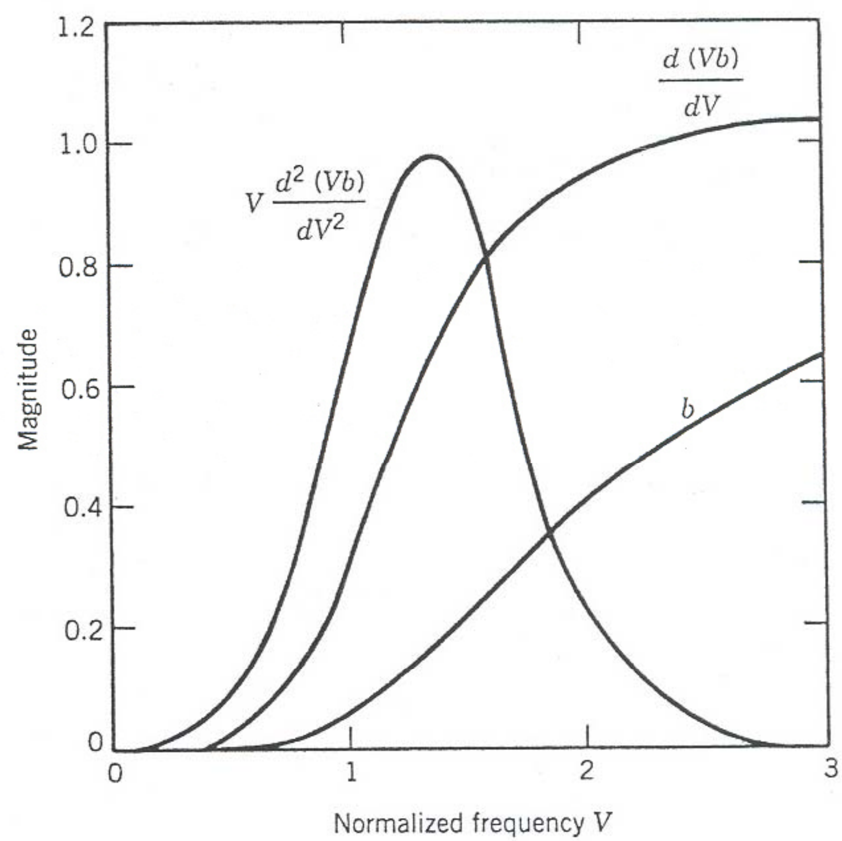


$\lambda_{zD} \sim 1.27\text{-}1.29\ \mu m$

Figure 2.8: Variation of refractive index *n* and group index n_g with wavelength for fused silica.

Waveguide Dispersion

- Waveguide dispersion D_W comes from the first and second derivatives of (Vb) with respect to V
- For the wavelength range considered, D_W is always negative.
- Therefore, sum of waveguide and material dispersion shifts zero-dispersio wavelength to a slightly longer wavelength



Waveguide dispersion

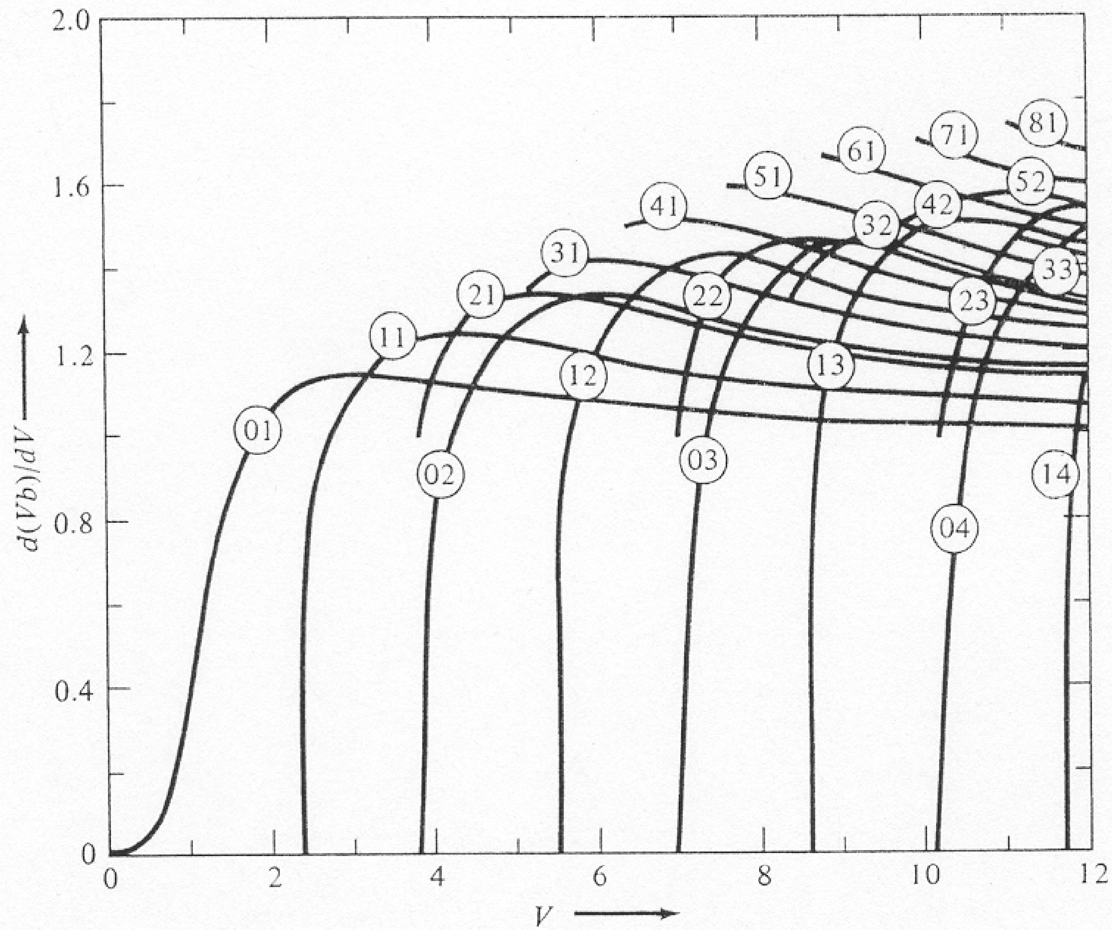
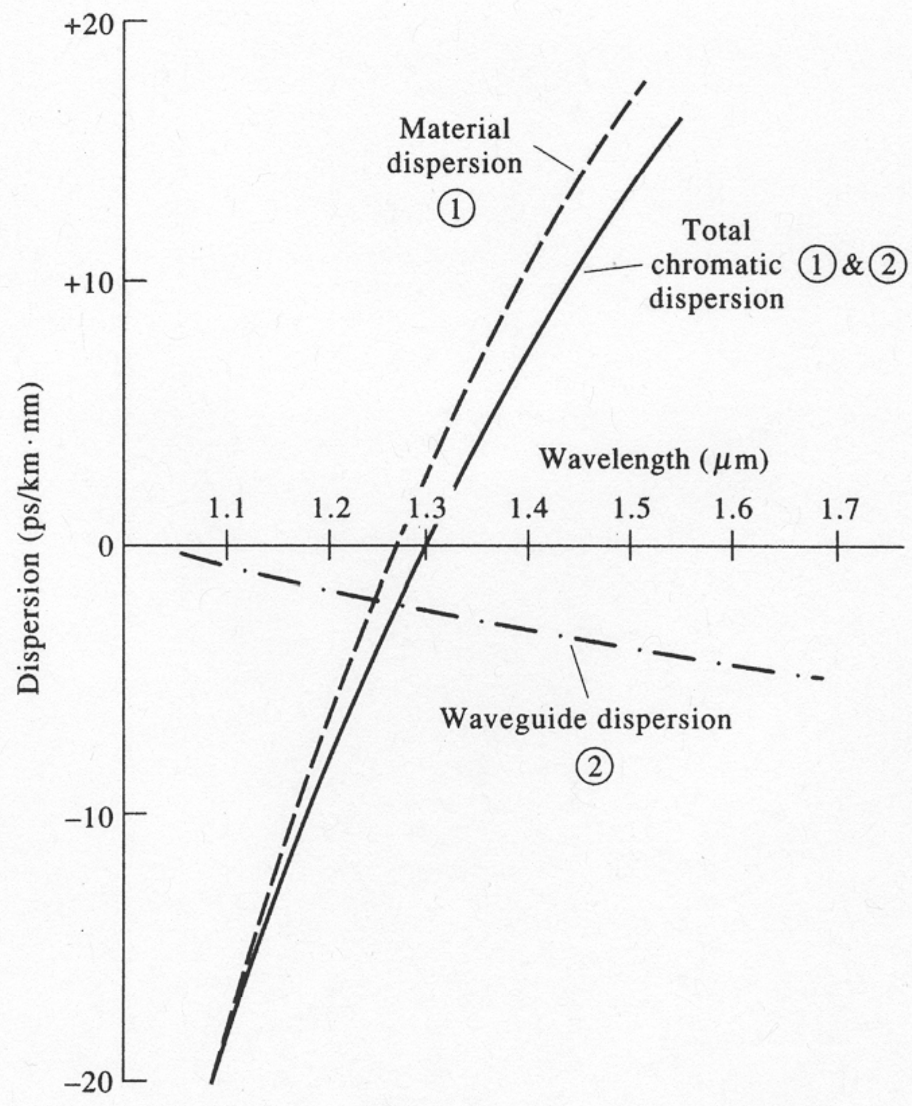


FIGURE 3-14
The group delay arising from waveguide dispersion as a function of the V number for a step-index optical fiber. The curve numbers jm designate the LP_{jm} modes. (Reproduced with permission from Gloge.[37])

Dispersion

$$\tau = DL\sigma$$

$$\Delta T = DL\Delta\lambda$$



(a)

D is dispersion parameter
L is the propagation length
σ is the spectral width

Dispersion (sum of material and waveguide dispersion)

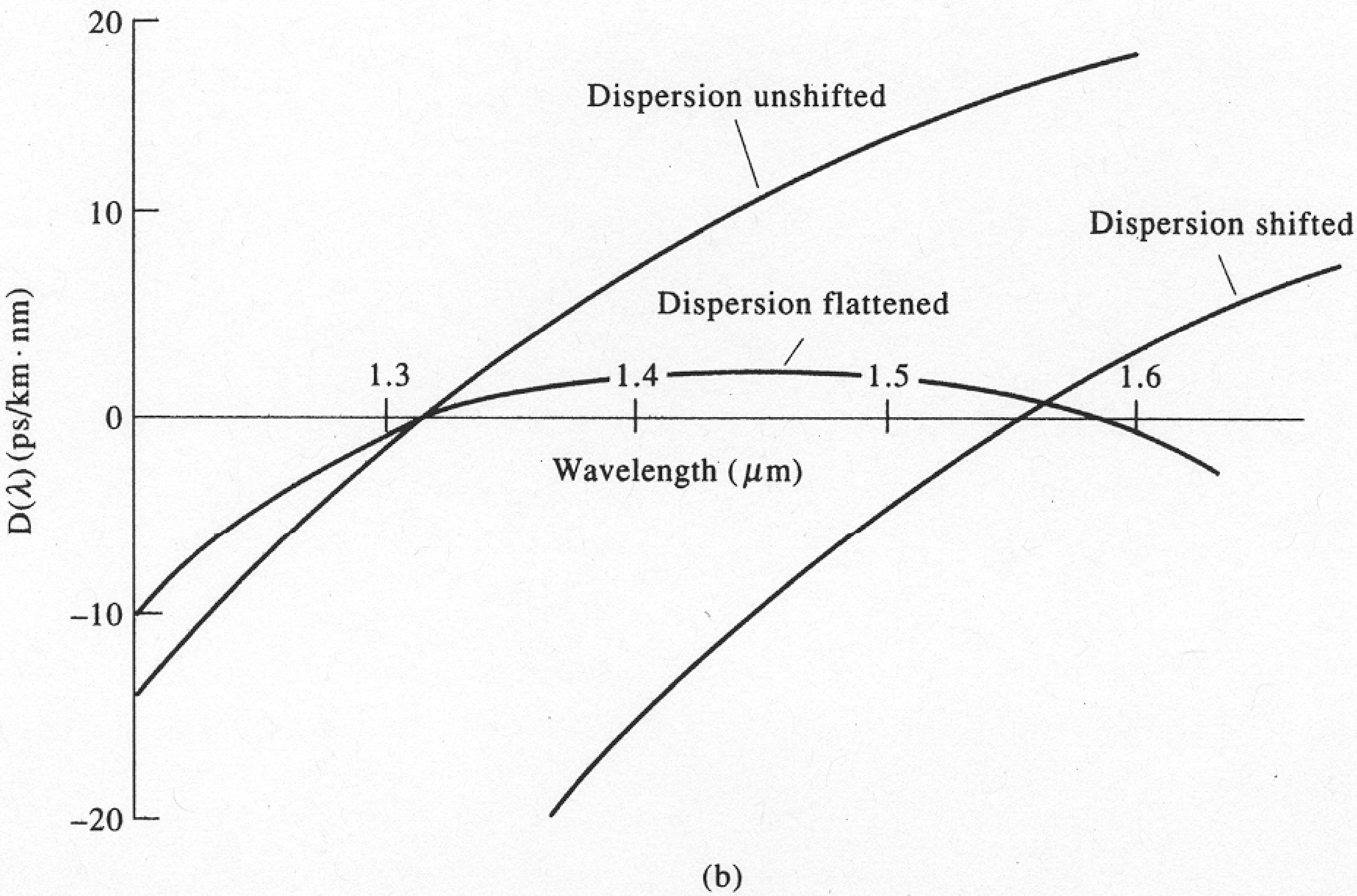


Figure 3-10 Group velocity disperion of (a) dispersion-unshifted 1.3 μm fiber and (b) dispersion- flattened and dispersion-shifted fibers. (After Reference [1].)

Dispersion Summary

- Single mode condition required for high performance
- Multimode fiber used for low cost
- Dispersion is designable.
- 1.3 micron: zero of dispersion
- 1.55 micron: minimum loss
- Zero dispersion is not good because of nonlinearities