SOLUTIONS

Name________________________________

Tear off and save the ID number below, to look up your grade later…

ECE 194d

"Midterm" Quiz

- <u>Extra pages are provided at back of exam, if needed.</u> **<u>Turn in all work!</u>**
- You are allowed one (1) single-sided sheet of notes.
- No calculators or other computing devices are allowed.

Good Luck!

Homework 4 – Extra Credit

Problem 1 - This problem involves the standard, 2-link, planar robot arm we've discussed several times in class.

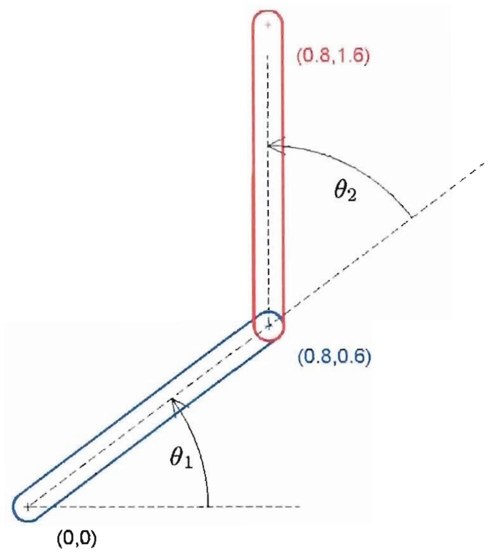


Fig. 1: *Link lengths are drawn above with correct scaling. Coordinates for "elbow" and end effector are labeled.*

a) **Solve for the Jacobian matrix** relating velocities of joints to those of the end effector, based on the given coordinates of the elbow (x_m=0.8, y_m=0.6) and end effector (x_e=0.8, y_e=1.6).
Recall that $\dot{\xi}_e = J\dot{q}_a$, and also $\tau_a = J^T F_e$. Here, $\xi = \begin{bmatrix} x_e \\ y_e \end{bmatrix}$ and $q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$.

$$J = \begin{bmatrix} -y_e\,, & -(y_e - y_m) \\ x_e\,, & (x_e - x_m) \end{bmatrix} = \begin{bmatrix} -1.6 & -1 \\ .8 & 0 \end{bmatrix}$$

Problem 2 – Lagrangian equations of motion: Point mass "marble", rolling along the surface of a funnel. (Equivalently, you can model the point-mass marble sliding along a frictionless funnel.)

a) For the system shown below, define a set of independent and complete generalized coordinates. (Hint, this problem is greatly simplified by assuming the ball has zero inertia!)

$$\boxed{\xi_1 = r \quad , \quad \xi_2 = \theta}$$

b) Derive the equation(s) of motion of the rolling ball. Gravity points downward in the -*h* direction (along the central axis of the funnel, pulling the ball downward). There are no non-conservative forces.

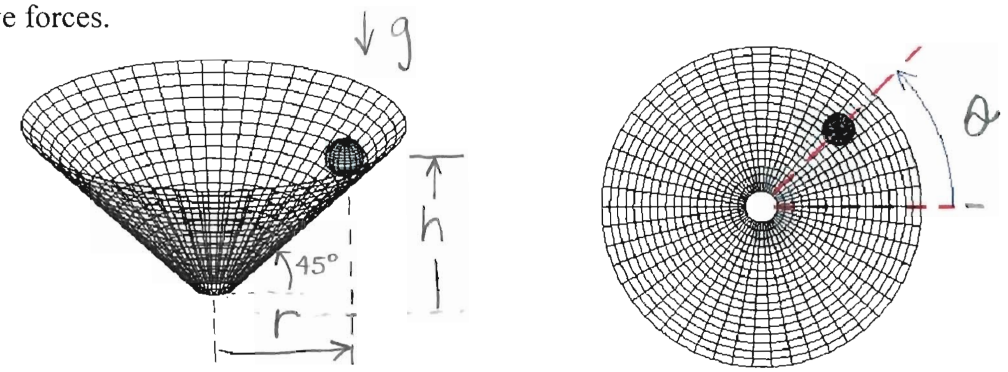


Fig. 2 – A ball rolls (or slides) in a "funnel". The ball has all mass concentrated in a point at its center, and its coordinates can be represented by r, θ, and h. The sides of the funnel are at 45 degrees, so that $h = r$ is a constraint.

$$T^* = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2 + \dot{h}^2\right), \quad \text{where } h = r$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\right)$$

$$= m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2$$

$$V = mgh = mgr$$

$$\mathcal{L} = T^* - V = m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - mgr, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_n}\right) - \frac{\partial \mathcal{L}}{\partial \xi_n} = \Xi_n$$

$$\xi_1 = r \rightarrow \frac{d}{dt}(2m\dot{r}) - (mr\dot{\theta}^2 - mg) = 0$$

$$2m\ddot{r} - mr\dot{\theta}^2 + mg = 0$$

$$\boxed{\ddot{r} = \frac{1}{2}(r\dot{\theta}^2 - g)} \leftarrow \text{Eq. 1}$$

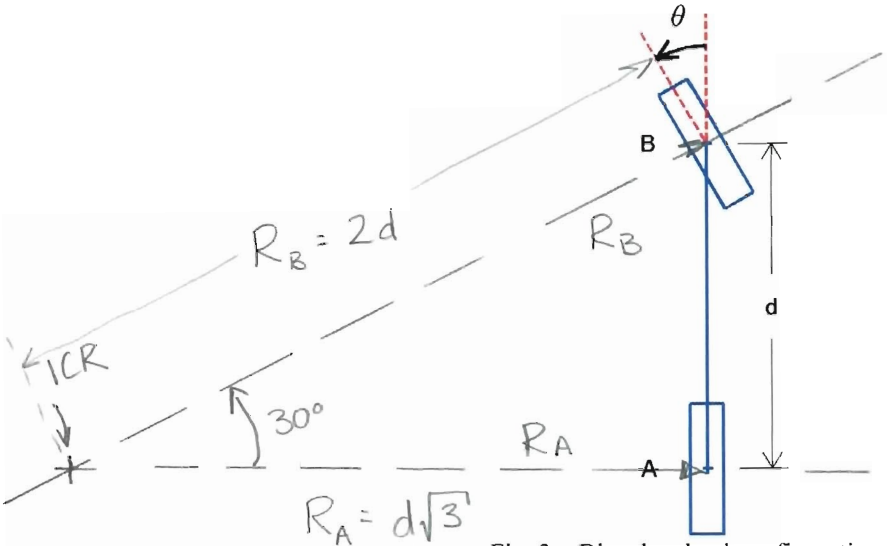
$$\xi_2 = \theta \rightarrow \frac{d}{dt}(mr^2\dot{\theta}) - (0) = 0$$

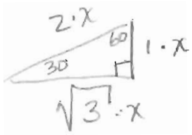
$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \rightarrow \boxed{\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}} \leftarrow \text{Eq. 2}$$

Problem 3 – Shown below is the wheel geometry for a 2-wheeled "bicycle", drawn to scale:

$$\theta = 30^{\circ}, \quad d = 1 \text{ meter.}$$

Each wheel can roll along its longer axis, but no slip is allowed perpendicular (sideways) to the rolling direction.





$$\sin(30^{\circ}) = \frac{d}{R_B}$$

Fig. 3 – Bicycle wheel configuration.

a) On the figure above, sketch the location of the instantaneous center of rotation (ICR).

b) What is the distance from the ICR to the center of wheel B?

$$R_B = d/\sin(30^{\circ}) = 2d = \boxed{2 \text{ meters}}$$

c) Assume that in this configuration, wheel A rolls at speed α and wheel B rolls at speed β. What is the ratio, $N = \frac{\alpha}{\beta}$, of the two wheel speeds?

$$N = \frac{\alpha}{\beta} = \frac{R_A}{R_B} = \frac{\sqrt{3}\, d}{2d} = \boxed{\frac{\sqrt{3}}{2}}$$

Problem 4 – Lagrangian equations of motion: Tricky suitcase dynamics.

In Fig. 4, a pendulum with a point mass, m_p, is mounted at an inner corner of a suitcase; the point mass is at a distance L_p from its pivot. θ_p is the absolute angle of the pendulum, measured CCW (counter-clockwise) from vertical.

The suitcase rests on an edge. It has a mass, m_s, and a moment of inertia, J_s. The center of mass is a distance R_s from the corner on which it rests, as shown. The absolute angle that the line (R_s) to the center of mass makes with respect to a vertical line is ϕ_s, as shown.

The suitcase has a motor that applies a torque, τ_{s2p}, from the suitcase to the pendulum.

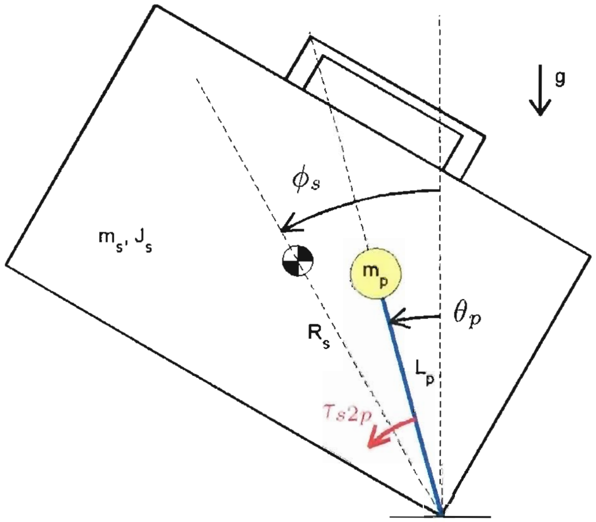
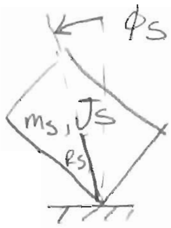


Fig. 4 – Massless seesaw with a (point-mass) cart driven on top.



$$T^*_s = \frac{1}{2}(m_s R_s^2 + J_s)\dot{\phi}_s^2$$

$$y_s = R_s \cos\phi_s$$

$$T^*_p = \frac{1}{2} m_p L_p^2 \dot{\theta}_p^2$$

$$y_p = L_p \cos\theta_p$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_n}\right) - \frac{\partial \mathcal{L}}{\partial \xi_n} = \Xi_n$$

a) Write an expression for the kinetic co-energy of the system, T^*.

$$\boxed{T^* = \frac{1}{2}(m_s R_s^2 + J_s)\dot{\phi}_s^2 + \frac{1}{2}(m_p L_p^2)\dot{\theta}_p^2}$$

b) Write an expression for the potential energy of the system, V.

$$\boxed{V = m_s g R_s \cos\phi_s + m_p g L_p \cos\theta_p}$$

c) Derive the equation(s) of motion.

$$\mathcal{L} = T^* - V = \frac{1}{2}(m_s R_s^2 + J_s)\dot{\phi}_s^2 + \frac{1}{2}(m_p L_p^2)\dot{\theta}_p^2 - g(m_s R_s \cos\phi_s + m_p L_p \cos\theta_p)$$

$$\xi_1 = \phi_s \rightarrow \frac{d}{dt}\left((m_s R_s^2 + J_s)\dot{\phi}_s\right) - (g m_s R_s \sin\phi_s) = -\tau_{s2p}$$

$$\boxed{(m_s R_s^2 + J_s)\ddot{\phi}_s - m_s g R_s \sin\phi_s = -\tau_{s2p}} \leftarrow \text{Eqn 1}$$

$$\xi_2 = \theta_p \rightarrow \boxed{(m_p L_p^2)\ddot{\theta}_p - m_p g L_p \sin\theta_p = +\tau_{s2p}} \leftarrow \text{Eqn 2}$$

Problem 5 – Rotational systems with gear ratios. (This problem should be easy…I hope!)

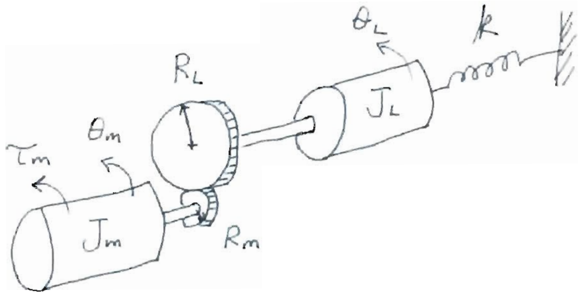


Fig. 5a – A motor driving a mechanical impedance.

For the system shown in Figure 5, all impedance elements are labeled. There is a small gear at the motor with radius R_m and which drives a larger gear with radius R_L, on the load end. (Each angle is positive when measured counter-clockwise, as shown.) Parameter values are:

$J_m = 0.1$ (kg*m^2) , $J_L = 0.5$ (kg*m^2) , $k = 48$ (N*m/rad) , $R_m = 1$ (cm) , $R_L = 5$ (cm)

a) What is the relationship (transfer function) between θ_m and θ_L ?

$R_m\theta_m = -R_L\theta_L$, or $\boxed{\dfrac{\theta_L}{\theta_m} = \dfrac{-R_m}{R_L}}$, Let $n \equiv \dfrac{-R_m}{R_L}$

b) How many degrees of freedom (required generalized coordinates) does the system have?

Just one! From "a", θ_L is always $\dfrac{-R_m}{R_L}\theta_m$.

c) What are the poles of the system?

We expected a pair of imaginary poles (on the $j\omega$-axis).

$$J_m\ddot{\theta}_m = \tau_m - n(J_L\ddot{\theta}_L + k\theta_L) \leftarrow \tau_{eff} = n\tau_L \qquad \theta_L = n\theta_m$$

$$J_m\ddot{\theta}_m = \tau_m - n^2(J_L\ddot{\theta}_m + k\theta_m)$$

where: $n = \dfrac{-R_m}{R_L} = \dfrac{-1}{5}$, $\boxed{n^2 = \dfrac{1}{25}}$ $\leftarrow \dfrac{1}{n^2} = 25$

$$J_m\ddot{\theta}_m + n^2(J_L\ddot{\theta}_m + k\theta_m) = \tau_m$$

$$\left(\frac{1}{n^2}\right)J_m\ddot{\theta}_m + J_L\ddot{\theta}_m + k\theta_m = \left(\frac{1}{n^2}\right)\tau_m$$

$$\left[\left(\frac{1}{n^2}J_m + J_L\right)s^2 + k\right]\Theta_m(s) = \left(\frac{1}{n^2}\right)T_m(S)$$

$$\frac{\theta_m}{\tau_m} = \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{J_m}{n^2} + J_L\right)s^2 + k} = \frac{25}{(2.5 + .5)s^2 + 4}$$

d) For extra credit (and perhaps to help in modeling), complete the block diagram (next page…).

char. eq: $3s^2 + 48 = 0$

$s^2 + 16 = 0$

$\boxed{s = \pm 4j}$ rad/sec

5d) (…continued.) EXTRA CREDIT:
To fully describe the dynamic system, determine what expression should go into each each of the 5 blank spaces (number #1 through #5) in the block diagram below. For #5, you simply need either a – sign or a + sign.

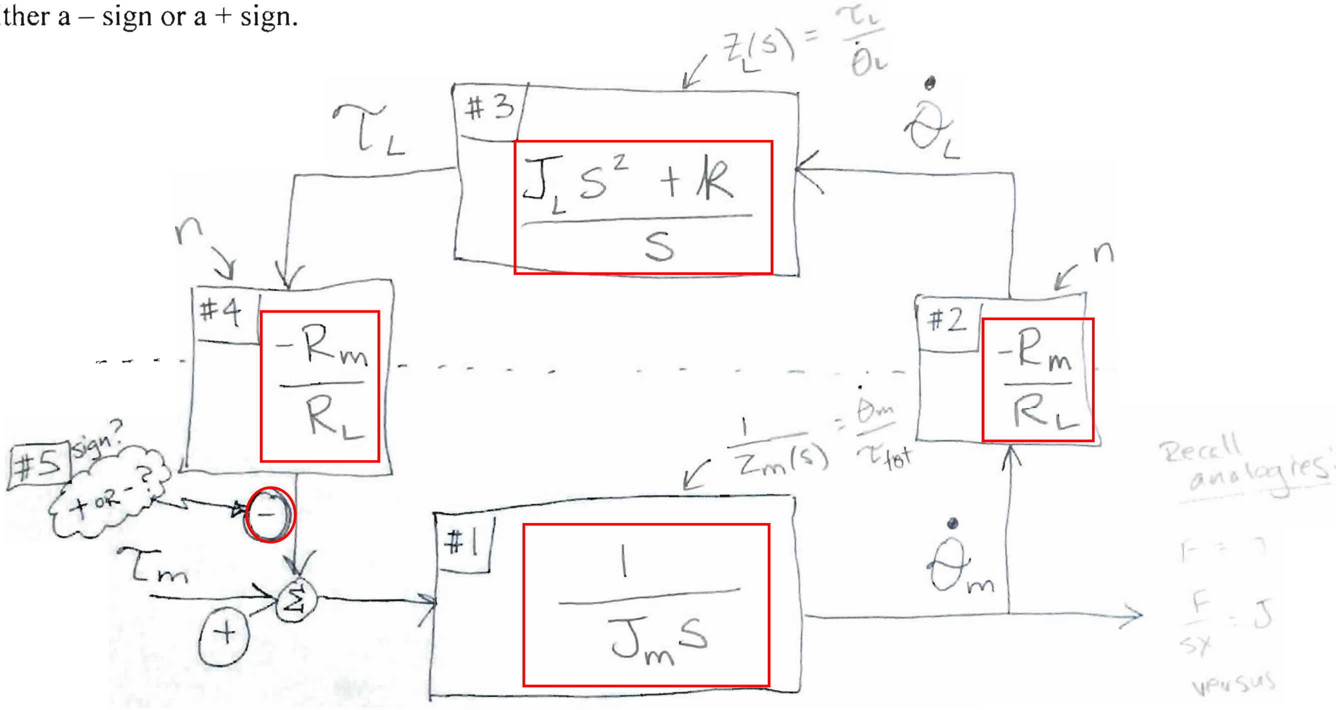


Figure 5b: Block diagram to be completed, for extra credit.

5e) Extra credit: Describe *briefly* the quantity in each of the boxes, #1-#4. (e.g., "a mechanical impedance", or 1/Z(s) [inverse impedance], or "gear ratio", etc…)

$$Z(s) = \frac{F(s)}{s \cdot X(s)} \quad \text{or} \quad \frac{\tau(s)}{s \cdot \theta(s)}$$

#1) is an "inverse" impedance, $\frac{1}{Z_m(s)}$;

#3) is a mechanical impedance, $Z_L(s)$.

#2 & #4) are both the same gear ratio.

#5) is negative, b/c passive mechanical elements give stable, negative feedba

(Blank page for extra work.)
Be sure to LABEL WHICH PROBLEM YOU ARE WORKING ON!

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