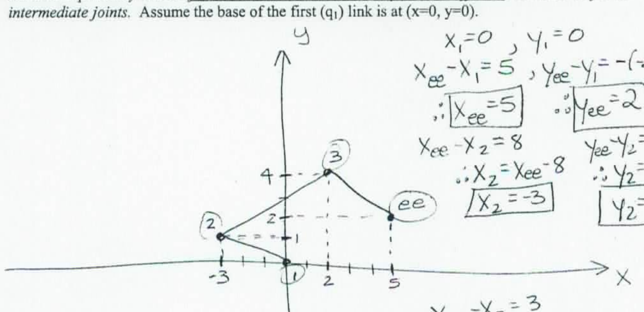
Problem 1 – Recall for a multi-link arm that $\dot{\xi}_e = J\dot{q}_a$, and also $\tau_a = J^T F_e$.

Here, $\xi_e = \begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix}$ gives the position and orientation of the end effector (at end-most tip of the last link), and $q_a = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ gives the *relative* angles of the links, such that $\phi_e = \theta_1 + \theta_2 + \theta_3$.

a) For a particular 3-link arm in a particular configuration, the Jacobian is as follows:
$J = \begin{bmatrix} -2 & -1 & 2 \\ 5 & 8 & 3 \\ 1 & 1 & 1 \end{bmatrix}$. **Sketch and <u>clearly label</u> the corresponding 3-link arm geometry.**
Hint: *It is probably easiest first to calculate the location of the end effector and later to fill in intermediate joints.* Assume the base of the first (q_1) link is at (x=0, y=0).



$x_1 = 0$, $y_1 = 0$

$X_{ee} - X_1 = 5$, $Y_{ee} - Y_1 = -(-2)$

$\therefore \boxed{X_{ee} = 5}$ $\therefore \boxed{Y_{ee} = 2}$

$X_{ee} - X_2 = 8$ $Y_{ee} - Y_2 = -(-1)$

$\therefore X_2 = X_{ee} - 8$ $\therefore Y_2 = Y_{ee} - 1$

$\boxed{X_2 = -3}$ $\boxed{Y_2 = 1}$

$X_{ee} - X_3 = 3$

$X_3 = X_{ee} - 3$

$\boxed{X_3 = 2}$

$Y_{ee} - Y_3 = -(2)$

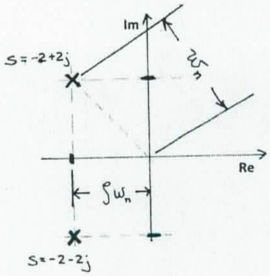
$Y_3 = Y_{ee} + 2$

$\boxed{Y_3 = 4}$

Clearly label the **COORDINATES OF THE END POINTS** of ALL 3 links in your sketch.
(The next page is BLANK to provide more space for needed calculations.)

Problem 2 – Control Law Partitioning for single-input single-output (SISO) systems.

a) Solve for the characteristic equation ($s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$) for a system with complex poles at $s = -2 \pm 2j$, as shown. (The picture of poles on the complex s-plane below may be helpful.)



$\boxed{s^2 + 2\zeta\omega_n s + \omega_n^2 = 0}$

$\zeta\omega_n = 2$

$\omega_n^2 = 2^2 + 2^2$

$= 4 + 4$

$= 8$

$\therefore \boxed{s^2 + 4s + 8 = 0} \leftarrow$ is the char. eqn.

For a simple model of a single-link arm, a control torque, τ, is to be used to regulate the angle (i.e., keep it near zero). Assume the dynamics of the system can be modeled as follows:

$$mL^2\ddot{\theta} = -B\dot{\theta} - mgL\sin\theta + \tau$$

b) Solve for a control law (i.e., an equation for control torque tau) for this system to cancel the natural system dynamics and to produce a closed-loop system that behaves like a linear, second-order dynamic system with poles at:

$$s = -2 \pm 2j$$

$\tau_{cancel} = B\dot{\theta} + mgL\sin\theta$

$\tau_{add} = mL^2(-4\dot{\theta} - 8\theta)$

$\boxed{\tau = \tau_{cancel} + \tau_{add} = (B - 4mL^2)\dot{\theta} - 8mL^2\theta + mgL\sin\theta}$

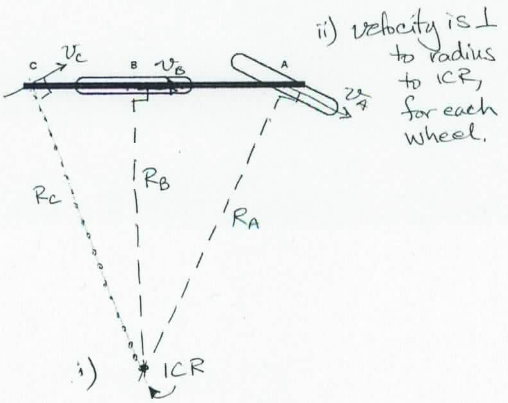
Problem 3 – The bike-trailer system.
The goal of this problem is to find the instantaneous center of rotation (ICR) for each of the 4 wheels in a system consisting of a bicycle pulling a two-wheeled trailer.

a) For the **bicycle alone, shown below**, points A, B, and C all exist on a single, rigid frame.
- i) Sketch the ICR for the bicycle, given the wheel positions and orientations shown.
- ii) Sketch the instantaneous directions of motion (velocity vectors) at points A, B, and C.
- iii) List, from lowest to highest, the instantaneous speed at points A, B, and C. E.g., <u>if</u> A were the slowest-moving of the three points and C were the fast point, then you would list:

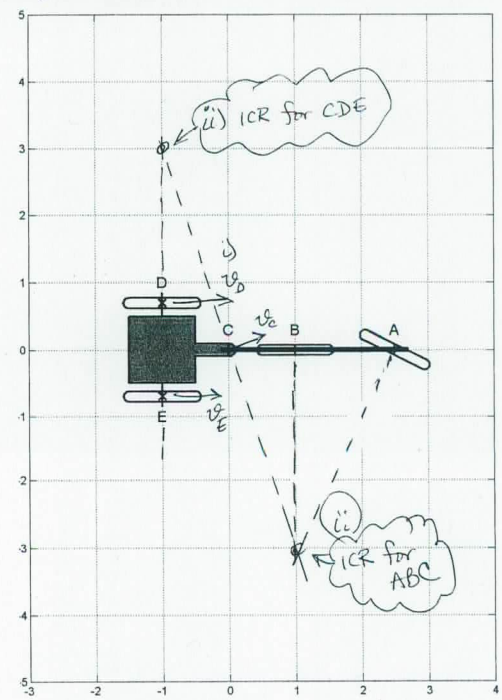
$v_A < v_B < v_C$.

$R_A > R_C > R_B$

iii) $\therefore \boxed{v_A > v_C > v_B}$

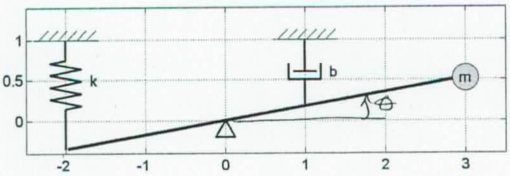


3.b) Below is the full 4-wheeled system: the bicycle is attached to a 2-wheeled trailer at point C. Points A, B, and C are part of the bicycle subsystem. Points C, D, and E are part of the trailer subsystem. Note that point C is a pivot connection between the bicycle and trailer, and so its instantaneous velocity must be compatible with the location of the ICR for the bicycle rigid-body subsystem and with the trailer rigid-body subsystem.
- i) Sketch the instantaneous directions of motion at points C, D, and E on the figure below.
- ii) Sketch **and clearly label** the ICR (instantaneous center of rotation) for each of the two rigid bodies: the bicycle and the trailer.



Problem 4 –Reflected Impedances: Mechanical elements connected via gear ratios.
For the see-saw system below, a rigid, massless bar can pivot about the apex of the small triangle. A spring, damper and mass are connected to the bar, as shown, each at different radii (r_k, r_b, and r_m) from the pivot point. Assume small angle motions (so linearization is valid).

a) Write the characteristic equation for the system, and solve for ω_n (natural frequency) and ζ (damping ratio, zeta), leaving the answer in symbolic form (in terms of *k*, *b*, *m*, r_k, r_b, and r_m).



(ignore gravity...)

$$(mL_m^2)\ddot{\theta} = \sum \tau = -L_b * F_b - L_k * F_k$$

$$= -L_b * (L_b\dot{\theta} \cdot b) - L_k(L_k\theta \cdot k)$$

$$mL_m^2\ddot{\theta} + bL_b^2\dot{\theta} + kL_k^2\theta = 0$$

$$\boxed{\omega_n = \left|\frac{L_k}{L_m}\right|\sqrt{\frac{k}{m}} \; ; \; \zeta = \frac{2\zeta\omega_n}{2\omega_n} = \frac{\left(\frac{bL_b^2}{mL_m^2}\right)}{2\left|\frac{L_k}{L_m}\right|\sqrt{\frac{k}{m}}} = \left|\frac{bL_b^2}{2L_kL_m\sqrt{k \cdot m}}\right|}$$

b) Now, solve for ω_n and ζ numerically, for m=1 (kg), k=900 (N/m), b=36 (N-s/m) and the geometry shown above. (Although you may not know r_k, r_b, and r_m exactly from the plot, you do know their values *relative* to one another, which is all that is needed to solve the problem.)
(Hint: numbers were picked so math works out "nicely" if you are correct, which may help you catch errors in part a…)

$$mL_m^2\ddot{\theta} + bL_b^2\dot{\theta} + kL_k^2\theta = 0$$

$$\left[(1)(3)^2s^2 + (36)(1)^2s + (900)(-2)^2\right]\theta = 0$$

$$9s^2 + 36s + 3600 = 0$$

$$s^2 + 4s + 400 = 0 \rightarrow \left\{ \begin{array}{l} \therefore \boxed{\omega_n = \sqrt{400} = 20} \\ 2\zeta\omega_n = 4 \\ \boxed{\zeta = \frac{4}{2\omega_n} = \frac{2}{20} = 0.1} \end{array} \right.$$