Homework 2 Solutions

Problem 1

Strogatz 2.5.5 (b)

Consider the following initial value problem:

\[ \dot{x} = \frac{|x|^{p/q}}{q}, \quad x(0) = 0 \]

Note that

\[ x(t) = 0 \quad \forall \ t \]

is a solution to the initial value problem (IVP) since

\[ \frac{dx}{dt} = 0 \quad \text{and} \quad |0|^{p/q} = 0 \]

In order to show that there is a unique solution to the IVP, we need to show that no other solution (other than \( x(t) = 0 \)) satisfies the IVP.

Separate the variables of the differential equation and integrate. Note that \( p \neq q \) since they have no common factors.

\[
\frac{dx}{dt} = |x|^{p/q} \\
\int dt = \int |x|^{-p/q} \, dx \\
t + C = \frac{1}{1 - p/q} |x|^{1-p/q} \text{sgn}(x)
\]

We can now find the particular solution using the initial condition, \( x(0) = 0 \). Evaluate the above expression for \( t = 0 \) and \( x = 0 \). Recall that \( p > q \), which means that \( 1 - p/q < 0 \).

\[ C = \frac{1}{1 - p/q} |0|^{1-p/q} \text{sgn}(0) \to -\infty \]

However, \( C \) must be finite. Thus, the only solution to this IVP is \( x(t) = 0 \).

2.6.1

While the simple harmonic oscillator \( m\ddot{x} = -kx \) oscillates in one dimension, the system is actually a two-dimensional (second order) system since two states are required to describe the dynamics of the system (one state for the position, \( x \), and one state for the velocity, \( \dot{x} \)). Thus, the text’s statement that “one-dimensional system can’t oscillate” is not violated.
Problem 2

Strogatz 2.5.6

Consider a water bucket with a hole in the bottom. Let $h(t)$ = height of the water remaining in the bucket at time $t$; $a$ = area of the hole; $A$ = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

Part (a)

By the conservation of mass, the amount of water leaving the bucket is equal to the change in the water volume in the bucket. The water volume in the bucket at time $t$ is

$$V(t) = Ah(t)$$

Thus, the change in the water volume in the bucket is

$$\dot{V}(t) = A\dot{h}(t)$$

Recall that $A$ is assumed to be constant. The amount of water leaving the bucket is

$$F(t) = av(t)$$

Since $F(t) = \dot{V}(t)$,

$$av(t) = A\dot{h}(t)$$

Part (b)

The change in potential energy is given by

$$\Delta PE = (\Delta m)gh = (\rho \Delta V)gh = (\rho A\Delta h)gh = (\Delta h)Apgh$$

where $\Delta m$ is the change in mass of the water in the bucket, $\rho$ is the density of the water, and $\Delta h$ is the amount by which the water in the bucket decreases.

The change in kinetic energy is given by

$$\Delta KE = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}(\rho A\Delta h) = \frac{1}{2}(\Delta h)Apv^2$$

The change in potential is equal to the kinetic energy. Thus,

$$(\Delta h)Apgh = \frac{1}{2}(\Delta h)Apv^2$$

Canceling appropriate terms, the above expression results in

$$v^2 = 2gh$$
Part (c)

Combining the results of Parts (a) and (c),
\[ av = A \dot{h} \]
\[ a^2 v^2 = A^2 \dot{h}^2 \]
\[ a^2 (2gh) = A^2 \dot{h}^2 \]
\[ \dot{h}^2 = 2gh \left( \frac{a}{A} \right)^2 \]
\[ \dot{h} = \pm \sqrt{2gh} \left( \frac{a}{A} \right) \]

Since a negative change in height corresponds to water flowing out of the bucket,
\[ \dot{h} = -\sqrt{2gh} \left( \frac{a}{A} \right) = -C \sqrt{h} \]

where \( C = \sqrt{2g} \left( \frac{a}{A} \right) \).

Part (d)

We are given the initial condition \( h(0) = 0 \) (an empty bucket at time \( t = 0 \)). We first find the general solution by separating the variables and integrating.
\[ \dot{h} = -C \sqrt{h} \]
\[ h^{-\frac{1}{2}} dh = -C dt \]
\[ \int h^{-\frac{1}{2}} dh = \int -C dt \]
\[ -C(t + t_0) = 2h^{\frac{1}{2}} \]

For \( h(0) = 0 \),
\[ -Ct_0 = 0 \]
\[ t_0 = 0 \]

Thus,
\[ -Ct = 2h^{\frac{1}{2}} \]
\[ h = \left( -\frac{1}{2} C t \right)^2 \]
\[ h = \frac{C^2}{4} t^2 \]

Thus, \( h(t) = \frac{C^2}{4} t^2 \) is a solution. Note that, physically, it does not make sense for \( h(t) > 0 \) for \( t > 0 \) since, once the bucket empties, it cannot refill on its own. Thus, we compose the following solution
\[ h(t) = \begin{cases} \frac{C^2}{4} (t - t_0)^2 & , t \leq t_0 \\ 0 & , t > t_0 \end{cases} \]

where \( t_0 \leq 0 \). The above solution is still continuous and still satisfies the IVP. This corresponds to a bucket that has emptied at time \( t = t_0 \leq 0 \) and stays empty for \( t > t_0 \). Thus, if we find the bucket empty at time \( t = 0 \), we cannot determine when that bucket had emptied since there are an infinite number of possibilities (it could have emptied at any time \( t_0 \leq 0 \)).
Problem 3

\[ \dot{x} = x^2 - 4x \]

Recall that

\[ \dot{x} = f(x) = -\frac{dV}{dx} \]

Thus,

\[
\int -dV = \int f(x)dx \\
\int -dV = \int x^2 - 4xdx \\
-\frac{1}{3}x^3 - 2x^2 + C \\
V = 2x^2 - \frac{1}{3}x^3 + C
\]

For simplicity, let \( C = 0 \). Thus,

\[ V(x) = 2x^2 - \frac{1}{3}x^3 \]

Figure 1: Plot of the potential function \( V(x) = 2x^2 - \frac{1}{3}x^3 \).

There is a local minimum at \( x = 0 \). Therefore, there is a stable fixed point at \( x^* = 0 \).

There is a local maximum at \( x = 4 \). Therefore, there is an unstable fixed point \( x^* = 4 \).

We can also verify these results by plotting the vector field for \( \dot{x} = x^2 - 4x \).
Figure 2: Plot of the vector field of $\dot{x} = f(x) = x^2 - 4x$. This plot confirms the results we obtained by plotting the potential function, $V(x)$. 
Problem 4

Strogatz 2.8.2

Figure 3: Slope field for $\dot{x} = f(x) = \sin(x)$. Trajectories are shown for various initial conditions.
Problem 5

Strogatz 2.8.3

\[ \dot{x} = -x, \quad x(0) = 1 \]

Part (a)

To solve the problem analytically, we separate the variables and integrate.

\[ \frac{dx}{dt} = -x \]

\[ \int x^{-1} \, dx = \int -dt \]

\[ \ln(x) = -t + C \]

\[ x = e^{-t+C} \]

Using the initial condition, we find the particular solution.

\[ x(0) = e^{0+C} = e^C = 1 \Rightarrow C = 0 \]

Thus, the solution to the IVP is

\[ x(t) = e^{-t} \]

The exact value at time \( t = 1 \) is

\[ x(1) = e^{-1} \approx 0.36787944117 \]

Part (b)

The Euler method uses the following update rule:

\[ x_{n+1} = x_n + f(x_n) \Delta t \]

For the given system, \( f(x_n) = -x_n \) and \( x_0 = 1 \). For \( \Delta t = 1 \),

\[ x_1 = x_0 + f(x_0) \Delta t \]

\[ = 1 + (-1)(1) \]

\[ = 1 - 1 \]

\[ x_1 = 0 \]

Thus,

\[ \hat{x}_e(1) = 0 \]

The remaining estimates (for different \( \Delta t \)s) will be calculated using MATLAB. The following table summarizes the results.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>10^0</th>
<th>10^{-1}</th>
<th>10^{-2}</th>
<th>10^{-3}</th>
<th>10^{-4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{x}(1) )</td>
<td>0</td>
<td>0.348678440100000</td>
<td>0.366032341273230</td>
<td>0.367695424770964</td>
<td>0.367861046432929</td>
</tr>
</tbody>
</table>
Part (c)

Figure 4: Plot of (a) the error \( E = |\hat{x}(1) - x(1)| \) as a function of \( \Delta t \) and (b) \( \ln(E) \) as a function of \( \ln(\Delta t) \).
Strogatz 2.8.4

Part (b)

The improved Euler method uses the following update rule:

\[
\begin{align*}
\tilde{x}_{n+1} &= x_n + f(x_n)\Delta t \\
x_{n+1} &= x_n + \frac{1}{2} [f(x_n) + f(\tilde{x}_{n+1})] \Delta t
\end{align*}
\]

For the given system, \( f(x_n) = -x_n \) and \( x_0 = 1 \). For \( \Delta t = 1 \),

\[
\begin{align*}
\tilde{x}_1 &= x_0 + f(x_0)\Delta t \\
&= 1 + (-1)(1) \\
\tilde{x}_1 &= 0 \\
x_1 &= x_0 + \frac{1}{2} [f(x_0) + f(\tilde{x}_1)] \Delta t \\
&= 1 + \frac{1}{2} [(-1) + (0)] (1) \\
&= 1 + \frac{1}{2}(-1) \\
x_1 &= \frac{1}{2}
\end{align*}
\]

Thus,

\[
\hat{x}_{ie}(1) = \frac{1}{2} = 0.5
\]

The improved Euler method results in a much more accurate estimate for \( x(1) \) than the Euler method. The remaining estimates (for different \( \Delta t \)s) will be calculated using MATLAB. The following table summarizes the results.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( 10^0 )</th>
<th>( 10^{-1} )</th>
<th>( 10^{-2} )</th>
<th>( 10^{-3} )</th>
<th>( 10^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{x}(1) )</td>
<td>0.5</td>
<td>0.36854098483552</td>
<td>0.367885618716192</td>
<td>0.367879502530691</td>
<td>0.367879441784619</td>
</tr>
</tbody>
</table>
Part (c)

Figure 5: Plot of (a) the error $E = |\hat{x}(1) - x(1)|$ as a function of $\Delta t$ and (b) $\ln(E)$ as a function of $\ln(\Delta t)$. 
Strogatz 2.8.5

Part (b)

The Runge-Kutta method uses the following update rule:

\[ k_1 = f(x_n)\Delta t \]
\[ k_2 = f(x_n + \frac{1}{2}k_1)\Delta t \]
\[ k_3 = f(x_n + \frac{1}{2}k_2)\Delta t \]
\[ k_4 = f(x_n + k_3)\Delta t \]
\[ x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \]

For the given system, \( f(x_n) = -x_n \) and \( x_0 = 1 \). For \( \Delta t = 1 \),

\[
\begin{align*}
  k_1 &= f(x_0)\Delta t = -x_0(1) = -1 \\
  k_2 &= f(x_0 + \frac{1}{2}k_1)\Delta t = f(1 - \frac{1}{2})(1) = -\frac{1}{2} \\
  k_3 &= f(x_0 + \frac{1}{2}k_2)\Delta t = f(1 - \frac{3}{4})(1) = -\frac{3}{4} \\
  k_4 &= f(x_0 + k_3)\Delta t = f(1 - \frac{1}{4})(1) = -\frac{1}{4} \\
  x_1 &= x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
       &= 1 + \frac{1}{6}\left(-1 - 1\left(\frac{1}{2}\right) - 2\left(\frac{3}{4}\right) - \frac{1}{4}\right) \\
       &= 1 + \frac{1}{6}\left(-\frac{15}{4}\right) \\
       &= 1 - \frac{5}{8} \\
       &= \frac{3}{8}
\end{align*}
\]

Thus,

\[ \hat{x}_{rK}(1) = \frac{3}{8} = 0.375 \]

The Runge-Kutta method results in a much more accurate estimate for \( x(1) \) than both the Euler method and the improved Euler method. The remaining estimates (for different \( \Delta ts \)) will be calculated using MATLAB. The following table summarizes the results.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>( 10^0 )</th>
<th>( 10^{-1} )</th>
<th>( 10^{-2} )</th>
<th>( 10^{-3} )</th>
<th>( 10^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{x}(1) )</td>
<td>0.375</td>
<td>0.367879774412498</td>
<td>0.367879441202355</td>
<td>0.367879441171446</td>
<td>0.367879441171445</td>
</tr>
</tbody>
</table>
Figure 6: Plot of (a) the error $E = |\hat{x}(1) - x(1)|$ as a function of $\Delta t$ and (b) $\ln(E)$ as a function of $\ln(\Delta t)$.
Problem 6

Strogatz 3.1.1

\[
\dot{x} = 1 + rx + x^2
\]

The fixed points of the above system are found by finding the values of \( x \) for which \( \dot{x} = 0 \).

\[
1 + rx + x^2 = 0 \Rightarrow x^* = \frac{1}{2} \left( -r \pm \sqrt{r^2 - 4} \right)
\]

A bifurcation occurs when the following equation is satisfied:

\[
\left. \frac{df}{dx} \right|_{(x^*, r_c)} = 0
\]

Evaluating the above expression, we get

\[
r_c + 2x^* = 0 \Rightarrow r_c = -2x^*
\]

We now plug this value of \( r \) into the equation for \( x^* \) to find the fixed point(s) for which there is a bifurcation.

\[
x_c^* = \frac{1}{2} \left( -2x_c^* \pm \sqrt{4(x_c^*)^2 - 4} \right)
\]

\[
0 = 4(x_c^*)^2 - 4
\]

\[
(x_c^*)^2 = 1
\]

\[
x_c^* = \pm 1
\]

Plugging in these values of \( x_c^* \), we find the values of \( r, r_c \), for which there is a bifurcation.

\[
r_c = \pm 2
\]

The following is a summary of the fixed points for different values of \( r \) and their stabilities.

- \( r < -2 \): Two fixed points

\[
x^* = -\frac{1}{2} \left( r + \sqrt{r^2 - 4} \right)
\]

\[
x^* = -\frac{1}{2} \left( r - \sqrt{r^2 - 4} \right)
\]

Note that \( \sqrt{r^2 - 4} \leq \sqrt{r^2} = |r| \). Thus, both of the above fixed points are positive.
\[ f(x) = 1 + rx + x^2 \]

Figure 7: Vector field for \( r < -2 \). There are two fixed points: the one with a smaller magnitude is stable, while the one with the larger magnitude is unstable.

- \( r = -2 \): Two repeated fixed point
  \[ x^* = 1 \]

Figure 8: Vector field for \( r = -2 \). The (repeated) fixed points are half-stable.

- \(-2 < r < 2\): No fixed points
  In this case \( r^2 - 4 < 0 \), thus there are no real values of \( x^* \) for which \( \dot{x} = 0 \).
Figure 9: Vector field for $-2 < r < 2$. There are no fixed points in this case.

- $r = 2$: Two (repeated) fixed points
  
  $$x^* = -1$$

Figure 10: Vector field for $r = 2$. The (repeated) fixed points are half stable.

- $r > 2$: Two fixed points
  
  $$x^* = -\frac{1}{2} \left( r + \sqrt{r^2 - 4} \right)$$
  
  $$x^* = \frac{1}{2} \left( r + \sqrt{r^2 - 4} \right)$$

Note that $\sqrt{r^2 - 4} \leq \sqrt{r^2} = |r|$. Thus, both of the above fixed points are negative.
Figure 11: Vector field for $r > 2$. There are two fixed points: the one with a smaller magnitude is unstable, while the one with a larger magnitude is stable.

There is a saddle-node bifurcation since two fixed points “collide” and then disappear at the bifurcation points ($r_c = -2$ and $r_c = 2$).

Figure 12: Bifurcation diagram for $\dot{x} = f(x) = 1 + rx + x^2$. 
The fixed points of the above system are found by finding the values of $x$ for which $\dot{x} = 0$.

$$rx + x^2 = 0 \Rightarrow x^* = 0, -r$$

Thus, we always have a fixed point at $x^* = 0$, and a bifurcation occurs at $x^*_c = 0$ when the following equation is satisfied:

$$\left. \frac{df}{dx} \right|_{(x^*_c, r_c)} = 0$$

Evaluating the above expression, we get

$$r_c + 2 \cdot x^*_c = 0 \Rightarrow r_c = 0$$

Thus, the qualitatively different vector fields are as follows:

- $r < 0$: Two distinct fixed points
  
  $x^* = 0$
  $x^* = -r > 0$

Figure 13: Vector field for $r < 0$. There are two fixed points: the fixed point at $x^* = 0$ is stable, while the positive fixed point is unstable.
- $r = 0$: Two repeated fixed points
  \[ x^* = 0 \]

![Figure 14: Vector field for $r = 0$. The (repeated) fixed points are half stable.](image)

- $r > 0$: Two distinct fixed points
  \[ x^* = 0 \]
  \[ x^* = r < 0 \]

![Figure 15: Vector field for $r > 0$. There are two fixed points: the fixed point at $x^* = 0$ is unstable, while the negative fixed point is stable.](image)

There is a transcritical bifurcation at $r_c = 0$ since there is always two fixed points, either repeated or distinct, that trade stabilities as they pass the bifurcation point.
Figure 16: Bifurcation diagram for $\dot{x} = f(x) = rx + x^2$. 