NONLINEAR PHENOMENA: ECE 183 / ME 169 / PHYS 106

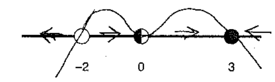
Name SOLUTIONS

Midterm Exam

May 2, 2013

______	Problem 0 (5 points) : Turn in one-page cheat sheet.
______	Problem 1 (10 points)
______	Problem 2 (10 points)
______	Problem 3 (15 points)
______	Problem 4 (20 points)
______	Problem 5 (15 points)
______	Problem 6 (25 points)
______	Total (out of 100)

Problem 1



The figure above shows fixed points on the line for a 1D system.
a) Write a valid expression for $f(x)$ that agrees with the figure.
b) Sketch $f(x)$ and draw arrows along the x-axis to indication the direction of flow on the line.

a) $f(x) = -(x+2)x^2(x-3)$

b)

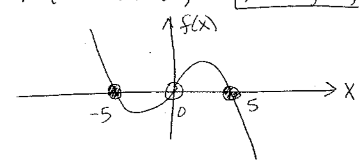
Problem 2

$f(x) = 25x - x^3$

a) Find all fixed points for f(x).
b) Sketch f(x), indicating fixed point locations and their stability.

a) $0 = x^*(25 - (x^*)^2) \rightarrow \boxed{x^* = 0, -5, 5}$

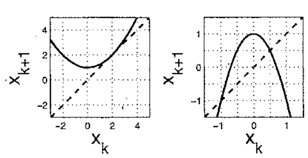
b)



(● stable, o unstable)

Problem 3

$P(x) = x_{k+1} = ax_k^2 + 1$



Above is a return map, $P(x)$, for a 1D discrete-time system (at left), with instances of $P(x)$ for two different values of a (right).

a) Write an expression to solve for the fixed points. (Do not bother to simplify this expression!)
b) Write an expression for the slope of $P(x)$.
c) For what values of a does the system have 1 stable and 1 unstable fixed point?

$\boxed{\text{a) } x^* = a(x^*)^2 + 1}$

$\boxed{\text{b) } \frac{dP}{dx} = 2ax}$

c) For slope $= -1$, $-1 = 2ax \rightarrow x = \frac{-1}{2a}$

$0 = a\left(\frac{-1}{2a}\right)^2 - \left(\frac{-1}{2a}\right) + 1$

$-1 = \frac{1}{4a} + \frac{2}{4a}$

$a = \frac{-3}{4}$

For slope $= +1$,

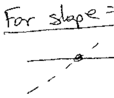
$-1 = \frac{1}{4a} - \frac{2}{4a}$

$a = +\frac{1}{4}$

$\boxed{a \neq 0}$ and $\boxed{\frac{-3}{4} < a < +\frac{1}{4}}$

to keep slope magnitude less than one

But: For slope $= 0$



← Only one (stable) fixed point!

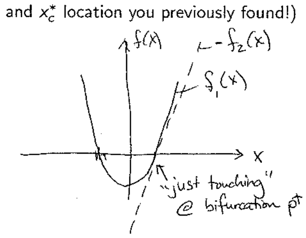
So...

Problem 4

$f(x) = f_1(x) - f_2(x) = x^4 - 1 + r(x-1)$
Define: $f_1(x) = x^4 - 1$
and: $f_2(x) = -r(x-1)$

a) Sketch $f_1(x)$, as a first step.
b) Now, find r_c and x_c^* at the point of bifurcation.
c) Identify the type of bifurcation.
d) Sketch the bifurcation diagram (x^* vs r). Do not try to capture the exact shape, but do clearly illustrate the directions and stability of all branches. (Take care that your bifurcation actually occurs at the r_c and x_c^* location you previously found!)

a)

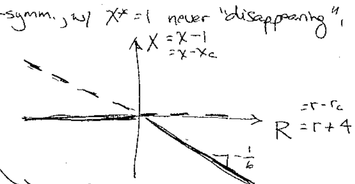


b) Both intersect (f_1 & $-f_2$) always @ $\boxed{x_c^* = 1}$

Slope $= 0 = (4x^3 + r)\big|_{x_c^*=1,\, r_c} = 4 + r_c$, $\therefore \boxed{r_c = -4}$

c) $\boxed{\text{Transcritical}}$, Non-symm., w/ $X^* = 1$ never "disappearing",

d) $x = x_c^* + X$
$r = r_c + R$



"bowl" shape, i.e. "$ax^2+bx+c=0$"...

Larger value unstable x^*

$(1+X)^4 - 1 + (-4+R)X = 0$

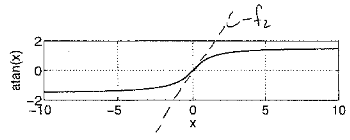
$(1 + 4X + 6X^2 + \cancel{4X^3} + \cancel{X^4}) - 1 - 4X + RX = 0$

H.O.T.

$6X^2 + RX = 0 \rightarrow X^* = 0, \frac{-R}{6}$

(keep x^2 & lower terms)

Problem 5



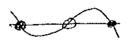
Above is a sketch of the arctangent function.
(Recall that $tan(\delta) \approx \delta$, for small delta.)

$f(x) = rx + \text{atan}(x)$

a) Solve for r_c and x_c^* at the bifurcation point.
b) Identify the type of bifurcation.
c) Sketch the bifurcation diagram. As in Problem 3: Do not try to capture the exact shape, but do clearly illustrate the directions and stability of all branches. (Take care that your bifurcation actually occurs at the r_c and x_c^* location you previously found!)

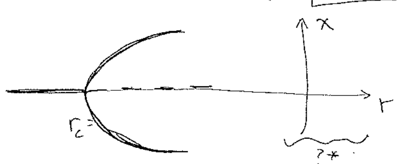
a) Always f_1 & $-f_2$ intersect @ $\boxed{x_c^* = 0}$

Matching slopes $\boxed{r_c = -1}$, since slope of $\text{atan}(x)\Big|_{x=0} = \underline{\underline{+1}}$

b)  $\boxed{\text{Supercritical Pitchfork}}$,

because 2 stable & 1 unstable x^* values.

c) 3 branches for lower magn. (but negative) r values: 3 branches for $\boxed{r_c < r < 0}$



[* Note: Just describes behavior locally, near r_c. (r>0 has only 1 solution.)]

Problem 6

$$\mathbf{A} = \begin{bmatrix} -3 & 5 \\ 1 & 1 \end{bmatrix}$$

a) Find eigenvalues and eigenvectors of **A**.
b) Write the linear dynamic equations for the 2 states, x and y.
c) On the axes below, sketch the eigenvectors as solid lines.
d) Then, sketch nullclines (i.e., $\dot{x} = 0$ and $\dot{y} = 0$) as dashed lines.
e) And sketch flow pattern (sample trajectories) in 2D phase space.

a) $\tau = -3+1 = -2$, $\Delta = (-3)(1) - (5)(1) = -8$

$\lambda_1 = \frac{1}{2}\left(-2 + \sqrt{(-2)^2 - 4(-8)}\right) = \frac{1}{2}(-2+6) = \frac{4}{2}$

$\boxed{\lambda_1 = +2}$ $\quad \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad [(-3-2), 5]\begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = \frac{1}{2}(-2-6)$

$\boxed{\lambda_2 = -4}$ $\quad \vec{V}_2 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $\qquad [(-3+4), 5]\,\vec{V}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) $\Rightarrow$ $\boxed{\begin{aligned} \dot{x} &= -3x + 5y \\ \dot{y} &= x + y \end{aligned}}$

d) $\dot{x} = 0 = -3x + 5y$

$\boxed{y = \frac{3}{5}x} \Leftarrow \dot{x}=0$ nullcline

$0 = x + y$

$\boxed{y = -x} \Leftarrow \dot{y}=0$ nullcline