Problem 1

The figure above shows fixed points on the line for a 1D system.

a) Write a valid expression for \( f(x) \) that agrees with the figure.
b) Sketch \( f(x) \) and draw arrows along the x-axis to indicate the direction of flow on the line.

\[ f(x) = \begin{cases} x + 2 & \text{for } x < 3 \\ x - 3 & \text{for } x \geq 3 \end{cases} \]

Problem 2

\( f(x) = 25x - x^2 \)

a) Find all fixed points for \( f(x) \).
b) Sketch \( f(x) \), indicating fixed point locations and their stability.

\[ \begin{align*}
\text{a)} & \quad \text{Fixed Points: } & x^* = & 0, 5, 5 \\
\text{b)} & \quad \text{Stability: } & & \text{Stable}
\end{align*} \]

Problem 3

\[ P(x) = x_{k+1} = x_k^2 + 1 \]

Above is a return map, \( P(x) \), for a 1D discrete-time system (at left), with instances of \( P(x) \) for two different values of \( a \) (right).

a) Write an expression to solve for the fixed points. (Do not bother to simplify this expression!)
b) Write an expression for the slope of \( P(x) \).
c) For what values of \( a \) does the system have 1 stable and 1 unstable fixed point?

\[ \begin{align*}
\text{a)} & \quad \text{Fixed Points: } & x^* = & \frac{-1 \pm \sqrt{1 - 4a}}{2a} \\
\text{b)} & \quad \text{Slope: } & \frac{dP}{dx} = & 2ax \\
\text{c)} & \quad \text{For slope } = -1, \quad -1 = 2ax \rightarrow x = \frac{-1}{2a} \\
\text{At } a = \frac{1}{4}, & \quad x = \frac{1}{2}, \quad \text{stable}
\end{align*} \]

Problem 4

\( f(x) = h(x) - \mu(x) = x^3 - 1 + r(x - 1) \)

Define: \( h(x) = x^3 - 1 \)

a) Sketch \( h(x) \) as a first step.
b) Now, find \( x^c \) and \( x^p \) at the point of bifurcation.
c) Identify the type of bifurcation.
d) Sketch the bifurcation diagram (\( x^c \) vs \( r \)). Do not try to capture the exact shape, but do clearly illustrate the directions and stability of all branches. (Take care that your bifurcation actually occurs at the \( r_c \) and \( x^c_c \) location you previously found!)

\[ \begin{align*}
\text{a)} & \quad \text{Bifurcation Point: } & x^c = & \frac{1}{4} \\
\text{b)} & \quad \text{Slopes: } & S_{\pm} = & \pm \frac{1}{4} \\
\text{c)} & \quad \text{Non-symmetric at } & x^c = & \text{disappearing, } \frac{1}{2} \\
\text{d)} & \quad \text{Bifurcation Shape: } & r_c = & \frac{1}{4}
\end{align*} \]
Above is a sketch of the arctangent function. (Recall that \( \tan(\delta) = \delta \), for small \( \delta \).)

\[ f(x) = \pi + \tan(x) \]

a) Solve for \( r_1 \) and \( r_2 \) at the bifurcation point.
b) Identify the type of bifurcation.
c) Sketch the bifurcation diagram. As in Problem 3: Do not try to capture the exact shape, but do clearly illustrate the directions and stability of all branches. (Take care that your bifurcation actually occurs at the \( r_1 \) and \( r_2 \) location you previously found!)

- **Always** \( r_1, r_2 \) intersect at \( x^* = 0 \)
- Matching slopes \( F_c = -1 \), since slope of \( c = \lambda(x) \) at \( y = 0 \)
- **Supercritical Pitchfork**
- because 2 stable + 1 unstable \( x^* \) values.

- **3 branches for stable node** (but negative)
- \( r \) values: 3 branches for \( r \leq r < 0 \)

**Note:** This describes behavior locally, near \( r_0 \) (or \( y > 0 \) has only 1 stable).