

Midterm Review

Tuesday, April 30, 2013

1D Systems

Fixed Points and Stability

Difference Equations to Vector Fields

Given a difference equation

$$\dot{x} = f(x)$$

we can plot the vector field, given by $f(x)$.

Example

$$\dot{x} = -\sinh(x) = -\frac{e^x - e^{-x}}{2}$$

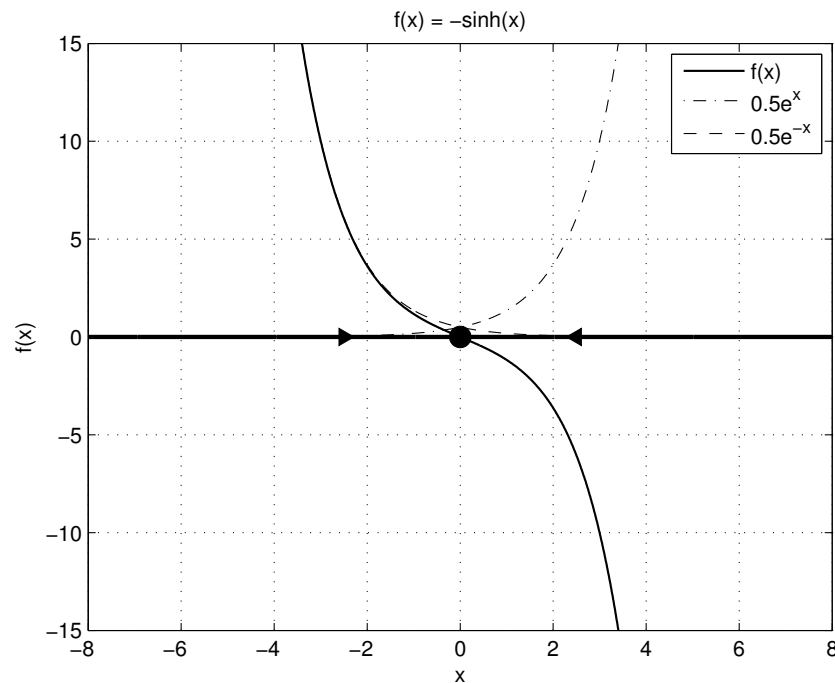


Figure 1: Plot of the vector field, $\dot{x} = f(x) = -\sinh x$. There is a fixed point at $x = 0$ since this is where $f(x)$ crosses the x -axis. Note that the slope at the fixed point, $x^* = 0$, is negative. Thus, the fixed point is stable.

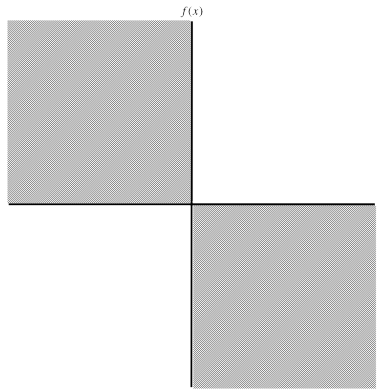


Figure 2: This plot shows the regions through which the function $f(x)$ must pass in order to have a stable fixed point. The cross-section of the two “axes” is the location of the fixed point, not necessarily at $x = 0$.

Vector Fields to Difference Equations

Given a vector field, we can systematically determine continuous equations that are consistent with it. Note that, if there is a continuous equation that is consistent with the vector field, there will be infinitely many continuous equations that also satisfy the conditions.

It is important to note that, if the equation is continuous, the stabilities of the fixed points on the vector field are constrained such that the flows are in agreement between two adjacent fixed points. For example, a continuous equation cannot have two adjacent unstable (or stable) fixed points.

Example

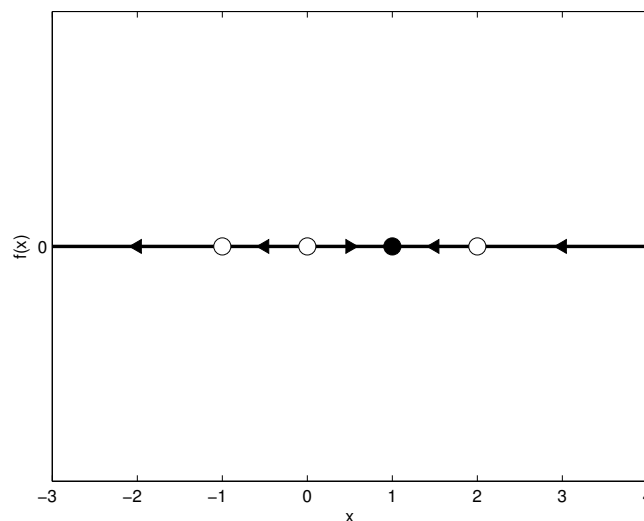


Figure 3: Find an (continuous) equation that is consistent with the above vector field. (Note that the fixed points at $x = -1$ and $x = 2$ are both right-half stable.)

It is useful to consider what sign the function value has for large magnitudes of x . For example, as $x \gg 2$, $f(x)$ has a negative value. This implies that there is a negative constant in front of the factored roots of $f(x)$.

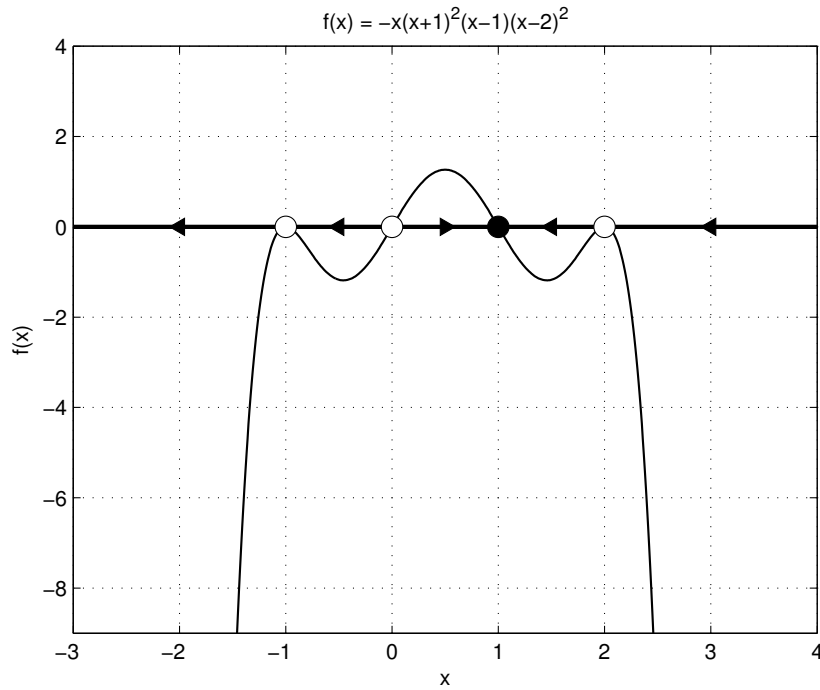


Figure 4: One such equation is $\dot{x} = -x(x + 1)^2(x - 1)(x - 2)^2$.

Potential Fields

Potential fields are another way of visualizing a 1D (continuous-time) system, and can be used to find its fixed points (and their stabilities). The relationship between the potential and the vector field is as follows:

$$f(x) = \dot{x} = -\frac{dV}{dx}$$

Thus, at a fixed point (i.e. when $\dot{x} = 0$), the derivative of the potential is zero.

When a fixed point is stable (i.e. when the slope of the vector field at the fixed point is negative), the second derivative of the potential is positive. Thus, there is a stable fixed point at local minimums of the potential. Conversely, when a fixed point is unstable (i.e. when the slope of the vector field at the fixed point is positive), the second derivative of the potential is negative. Thus, there is an unstable fixed point at local maximums of the potential.

Note: If a function has a critical point for which its derivative is zero and the second derivative is positive at this point, then the function has a local minimum at the critical point. If, however, the function has a critical point for which its derivative is zero and the second derivative is negative at this point, then f has local maximum at the critical point.

Example

$$\dot{x} = -\sinh(x) = -\frac{e^x - e^{-x}}{2}$$

Find the equation for the potential.

$$\begin{aligned} -\frac{dV}{dx} &= -\sinh(x) \\ \int dV &= \int \sinh(x) dx \\ \int dV &= \int \frac{1}{2} (e^x - e^{-x}) dx \\ V &= \frac{1}{2} (e^x + e^{-x}) + C \end{aligned}$$

The integration C only shifts the plot up or down and thus does not affect the locations of the minimums and maximums. Therefore we can pick an arbitrary value for C . Let us choose $C = 0$ for convenience.

$$V(x) = \frac{1}{2} (e^x + e^{-x})$$

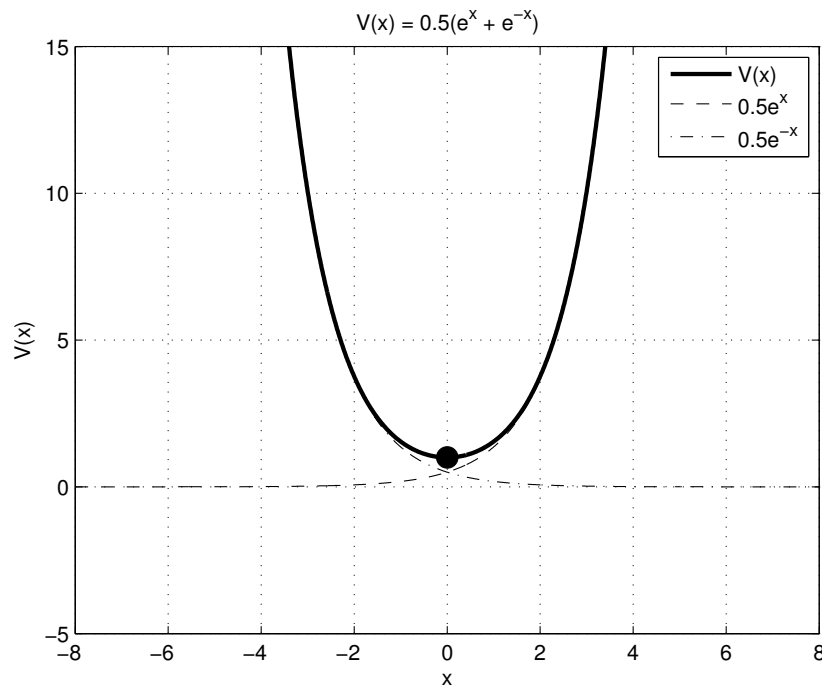


Figure 5: Plot of the potential field, $V(x) = \frac{1}{2} (e^x + e^{-x})$. There is a local minimum at $x = 0$, indicating that there is a stable fixed point there. This supports our results from plotting the vector field to determine the fixed point(s).

Poincaré Maps

The Poincaré Map, $P(x_k) = x_{k+1}$, defines where an initial state, $x_k \in S$, will be when it next passes through the Poincaré section S .

For a discrete system, the difference equation is the Poincaré Map. For Poincaré Maps, fixed points occur when the map passes through the same point of the Poincaré section as it did in the previous pass. Thus, a fixed point occurs where

$$P(x^*) = x^*$$

The fixed points and their stability are determined in a similar way as when plotting vector fields. While in the vector field case, fixed points occur when the vector field is equal to zero, fixed points occur when the Poincaré Map equals its argument (x_k). Thus, when finding fixed points, we can look for the points where the Poincaré Map crosses the “ $y = x$ ” line (or, in this case, $x_{k+1} = x_k$ line). In addition, while for vector fields, stability is determined by the sign of the slope at the fixed point, the stability of the fixed points when using Poincaré Maps is determined by the magnitude of the slope at the fixed points.

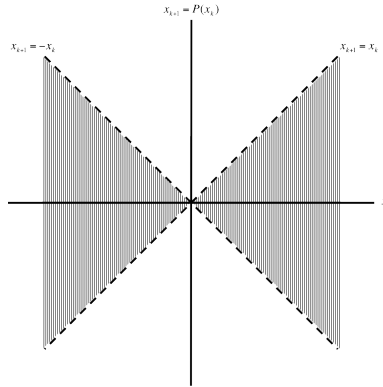


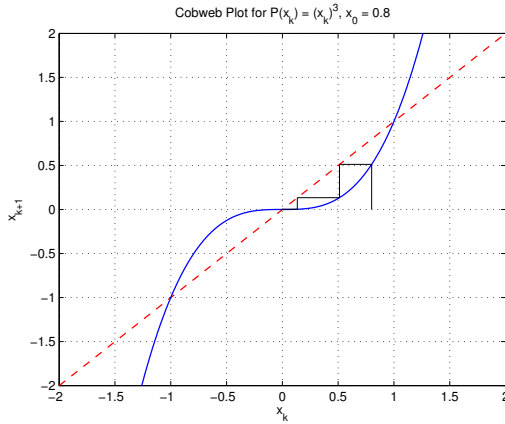
Figure 6: This plot shows the regions through which the Poincaré must pass in order to have a stable fixed point. The cross-section of the two “axes” is the location of the fixed point, not necessarily at $x_{k+1} = x_k = 0$. When the slope at a fixed point has a magnitude less than one, the fixed point is stable. If the magnitude of the slope is greater than one, the fixed point is unstable. If the magnitude is equal to one, higher order terms must be considered. A graphical approach can also be used to determine the stability.

Cobweb Plots

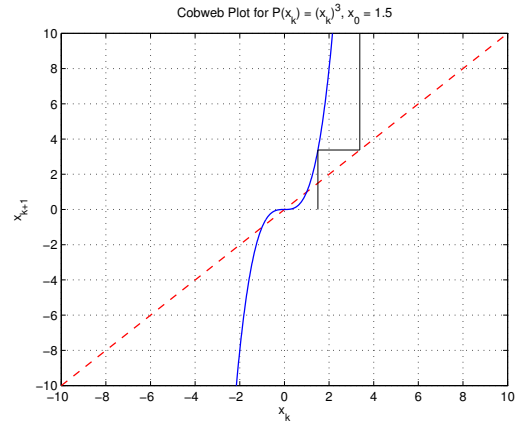
Cobweb plots show the points through which the Poincaré map passes through the section S as k increases. We start with an initial condition, x_0 , and, from the Poincaré map, determine the value of x_1 . If we continue this process of finding the next value using the Poincaré map, we can get a sense of the behavior of the system (e.g. initial conditions converge to a fixed point). Common behaviors include initial conditions converging to a fixed point, initial conditions going off to plus or minus infinity, and initial conditions entering oscillations (neither converging or diverging).

Example

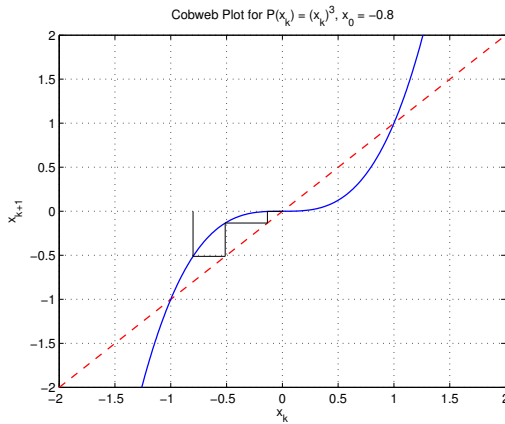
$$P(x_k) = x_{k+1} = (x_k)^3$$



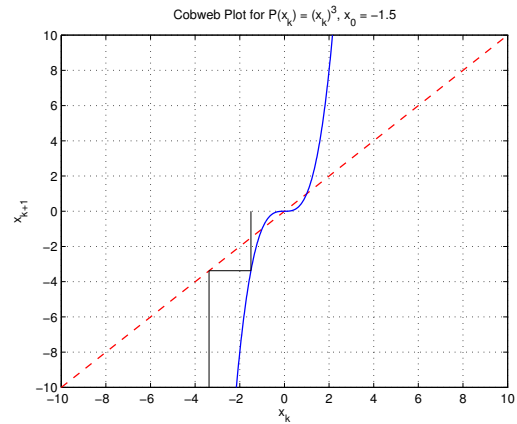
(a) Cobweb plot for $x_0 = 0.8$.



(b) Cobweb plot for $x_0 = 1.5$.



(c) Cobweb plot for $x_0 = -0.8$.



(d) Cobweb plot for $x_0 = -1.5$.

Figure 7: Poincaré Map for $P(x_k) = x_{k+1} = (x_k)^3$. Cobwebs are plotted for various initial conditions. Note that the magnitude of the slope at the fixed point at $x^* = 0$ is less than one, thus the fixed point is stable. Initial conditions $-1 < x_0 < 1$ converge to $x^* = 0$. The other two fixed points ($x^* = \pm 1$) are unstable since the magnitudes of their slopes are greater than one. Initial conditions $x > 1$ or $x < -1$ go off to plus infinity and minus infinity, respectively.

Bifurcations

We have covered four types of bifurcations in this class:

- **Saddle-node bifurcations**

The normal form of a saddle-node bifurcation is

$$\dot{x} = r \pm x^2$$

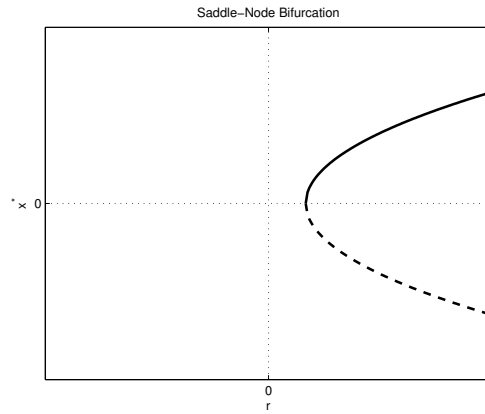


Figure 8: Basic shape of a saddle-node bifurcation. For saddle-node bifurcations, as r approaches the bifurcation point, the fixed points converge and then disappear after the bifurcation point.

- **Transcritical bifurcations**

The normal form of a transcritical bifurcation is

$$\dot{x} = rx \pm x^2$$

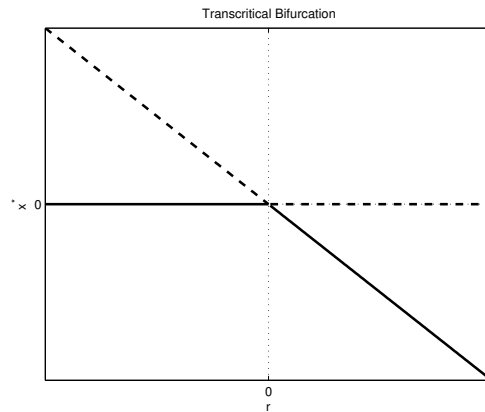


Figure 9: Basic shape of a transcritical bifurcation. For transcritical bifurcations, there are always two fixed points, either repeated or distinct. The trade stabilities after they pass the bifurcation point.

- **Subcritical pitchfork bifurcations**

The normal form of a subcritical pitchfork bifurcation is

$$\dot{x} = rx + x^3$$

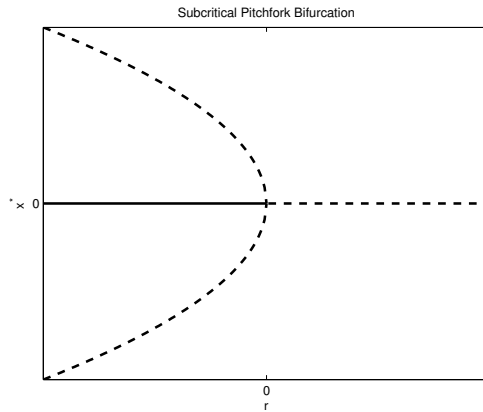


Figure 10: Basic shape of a subcritical pitchfork bifurcation. A subcritical pitchfork has one unstable fixed point on one side of the bifurcation point. That fixed point remains a fixed point for all values of r , but switches stability after passing the bifurcation point. Two other fixed points also appear after passing the bifurcation point, and they are both unstable.

- **Supercritical pitchfork bifurcations**

The normal form of a supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3$$

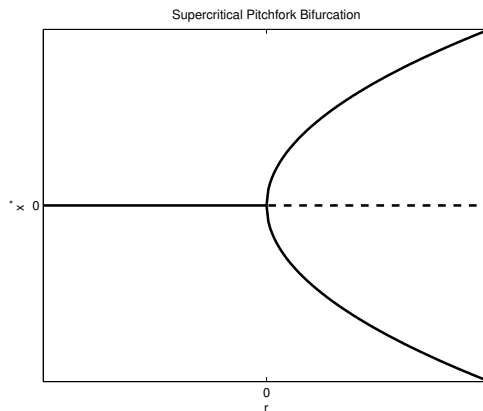


Figure 11: Basic shape of a supercritical pitchfork bifurcation. A supercritical pitchfork has one stable fixed point on one side of the bifurcation point. That fixed point remains a fixed point for all values of r , but switches stability after passing the bifurcation point. Two other fixed points also appear after passing the bifurcation point, and they are both stable.

Normal Forms

All bifurcations will look like their respective normal forms if we are looking closely enough around the bifurcation point. We can express a function in terms of its bifurcation type's normal form by approximating the function about the bifurcation point using the Taylor series expansion.

$$\begin{aligned}
f(x, r) = f(x_c^*, r_c) &+ \frac{\partial f(x_c^*, r_c)}{\partial x}(x - x_c^*) + \frac{\partial f(x_c^*, r_c)}{\partial r}(r - r_c) \\
&+ \frac{1}{2!} \frac{\partial^2 f(x_c^*, r_c)}{\partial x^2}(x - x_c^*)^2 + \frac{1}{2!} \frac{\partial^2 f(x_c^*, r_c)}{\partial r^2}(r - r_c)^2 \\
&+ \dots
\end{aligned}$$

Putting the function into a normal form can be useful when working with functions that we may not know how to plot very well. If we are just interested in the type of bifurcation we have, we can look at the normal form instead of plotting the vector field for various values of r .

2D Systems

For a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the parameters τ and Δ are given as follows

$$\begin{aligned}
\tau &= \text{trace}(A) = a + d \\
\Delta &= \det(A) = ad - bc
\end{aligned}$$

Its eigenvalues are given by

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Eigenvectors

When drawing phase plots, it is very helpful to have “guide” lines to help determine the shape of the plot. These “guide” lines include the eigenvectors and the nullclines.

Some phase plots will have **straight-line trajectories**. These straight-line trajectories occur along the eigenvectors of a system if those eigenvectors are real.

Nullclines

The nullclines are the curves for which either $\dot{x} = 0$ or $\dot{y} = 0$ for the 2D system with states x and y . The nullclines indicate where the flow is purely horizontal or vertical.

Types of Fixed Points

The following are the various types of fixed points for 2D systems:

- Stable and unstable nodes
 Stable: $\Delta > 0$, $\tau < 0$, and $\tau^2 - 4\Delta > 0$
 Unstable: $\Delta > 0$, $\tau > 0$, and $\tau^2 - 4\Delta > 0$

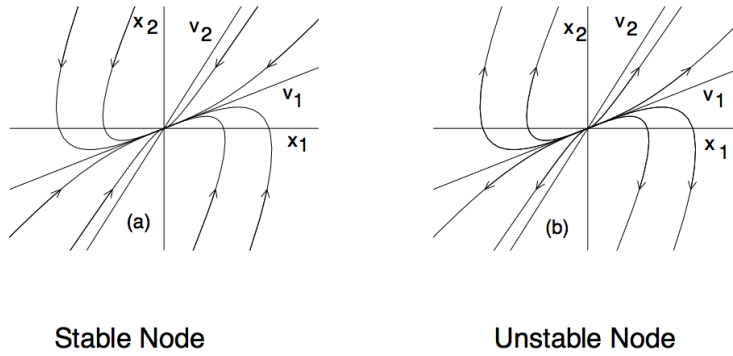


Figure 12: Khalil, H. *Nonlinear Systems and Control: Lecture #3: Second-Order Systems* [PDF Document]. Received from <http://www.egr.msu.edu/khalil/NonlinearSystems/Sample/>.

- Stable and unstable foci
 Stable: $\Delta > 0$, $\tau < 0$, and $\tau^2 - 4\Delta < 0$
 Unstable: $\Delta > 0$, $\tau > 0$, and $\tau^2 - 4\Delta < 0$

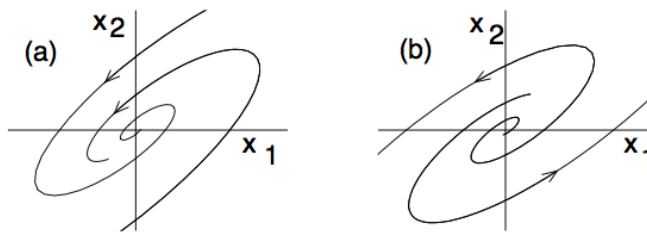
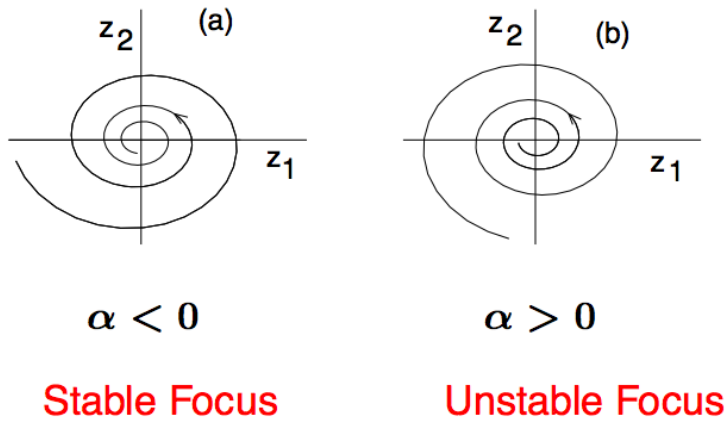
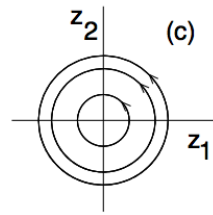


Figure 13: Khalil, H. *Nonlinear Systems and Control: Lecture #3: Second-Order Systems* [PDF Document]. Received from <http://www.egr.msu.edu/khalil/NonlinearSystems/Sample/>.

- Centers
 $\Delta > 0$ and $\tau = 0$



$$\alpha = 0$$

Center

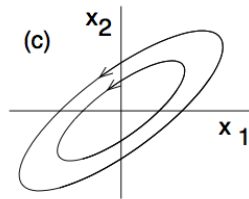
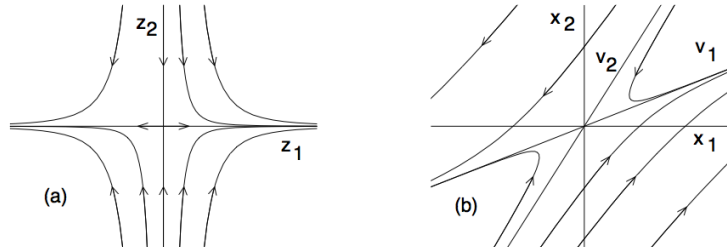


Figure 14: Khalil, H. *Nonlinear Systems and Control: Lecture #3: Second-Order Systems* [PDF Document]. Received from <http://www.egr.msu.edu/khalil/NonlinearSystems/Sample/>.

- Saddle points
 $\Delta < 0$



Phase Portrait of a Saddle Point

Figure 15: Khalil, H. *Nonlinear Systems and Control: Lecture #3: Second-Order Systems* [PDF Document]. Received from <http://www.egr.msu.edu/khalil/NonlinearSystems/Sample/>.

- Star (degenerate) nodes
 Stable: $\Delta > 0$, $\tau < 0$, and $\tau^2 - 4\Delta = 0$
 Unstable: $\Delta > 0$, $\tau > 0$, and $\tau^2 - 4\Delta = 0$
- Lines of fixed points
 Stable: $\Delta = 0$ and $\tau < 0$
 Unstable: $\Delta = 0$ and $\tau > 0$

Example

$$\dot{x} = 5x + 2y \quad , \quad \dot{y} = -17x - 5y$$

In vector form,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, our A matrix is

$$A = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix}$$

For this matrix,

$$\Delta = ad - bc = -25 + 34 = 9 > 0$$

$$\tau = a + d = 5 - 5 = 0$$

Thus, the fixed point is a center.

The eigenvalues of A are

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\sqrt{-4(9)}}{2} = \frac{\sqrt{-36}}{2} = j3$$

and

$$\lambda_2 = -j3$$

The eigenvector for $\lambda_1 = j3$ is found as follows

$$\begin{aligned} \begin{bmatrix} j3 - 5 & -2 \\ -17 & j3 + 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} (j3 - 5)v_1 - 2v_2 \\ 17v_1 + (j3 + 5)v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow v_2 &= \frac{1}{2}(j3 - 5)v_1 \Rightarrow 17 + \frac{1}{2}(j3 + 5)(j3 - 5)v_1 = 17v_1 - 17v_1 = 0 \Rightarrow v_1 = v_1 \\ \Rightarrow v &= \begin{bmatrix} 2 \\ j3 - 5 \end{bmatrix} \end{aligned}$$

The eigenvector for $\lambda_2 = -j3$ is just the complex conjugate of the eigenvector for $\lambda_1 = j3 = \lambda_2^*$.

Since the eigenvectors are complex, we cannot draw them on the phase portrait. However, we can still draw the nullclines. The nullclines are given by the following equations

$$\begin{aligned} \dot{x} = 5x + 2y = 0 &\Rightarrow y = -\frac{5}{2}x \\ \dot{y} = -17x - 5y = 0 &\Rightarrow y = -\frac{17}{5}x \end{aligned}$$

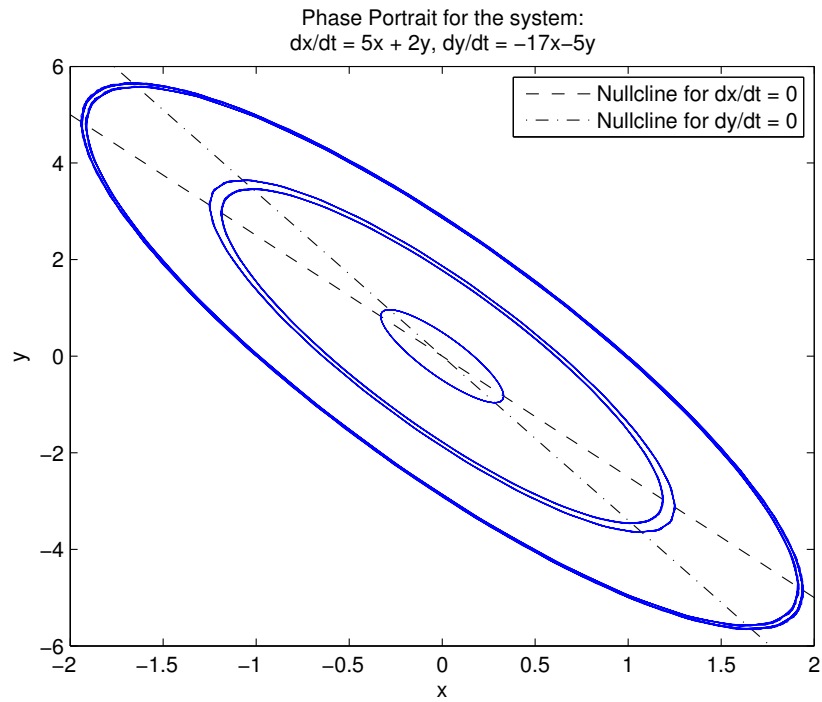


Figure 16: Phase portrait for the given system. While the eigenvectors cannot be drawn, the nullclines are plotted. The nullclines intersect the phase plot where either of the derivatives of the states is equal to zero.