Methods to show a closed orbit DOES exist:

1. Poincaré-Bendixson Theorem.
2. Lienard's Theorem.

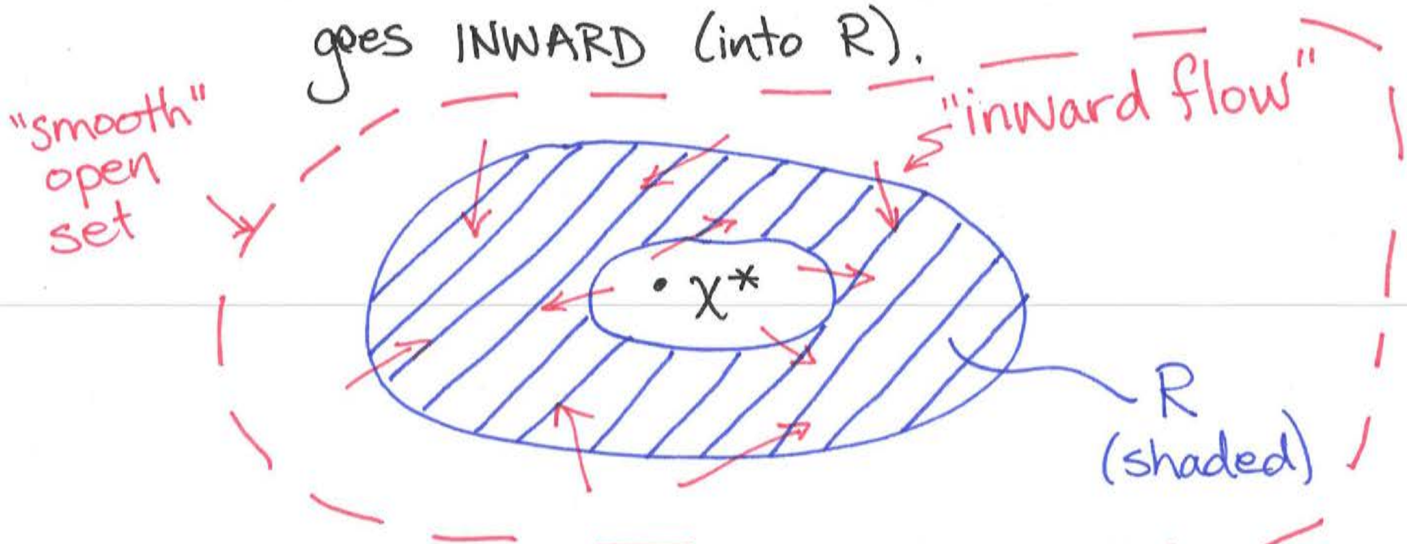
1. Poincaré-Bendixson Thm. (P-B Thm)

For a 2D system, $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in \mathbb{R}^2$, where $\vec{f}(\vec{x})$ is "smooth enough" over some open set, we look for the following conditions on some closed, bounded region R:

a) R is within the "smooth enough" set.

b) R contains NO FIXED POINTS.

c) Flow on all boundaries of R goes INWARD (into R).



→ Then R contains a closed orbit.

• Examples 7.3.2/7.3.3 (Strogatz)

Glycolysis. Biochemical process to break down sugar (for energy).

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

x: concentration of ADP

y: concentration glucose

(a, b → kinetic parameters)

- Let's try to demonstrate oscillations exist...

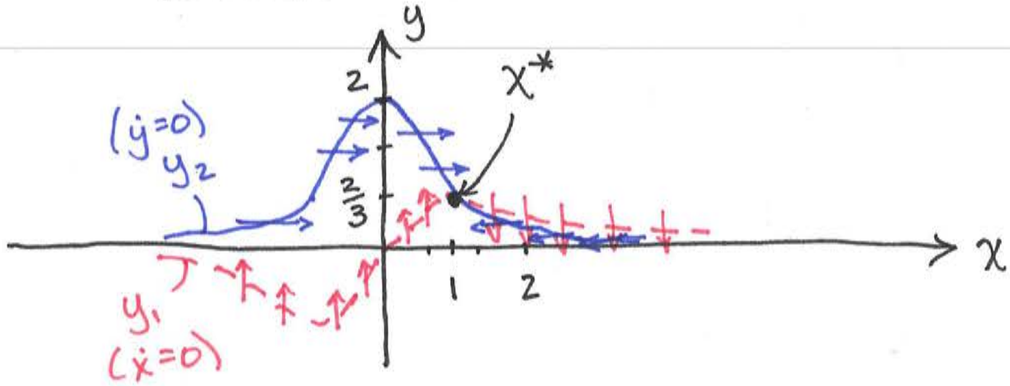
a) Nullclines: $\dot{x} = 0 = -x + ay + x^2 y \rightarrow \boxed{y_1 = \dfrac{x}{a+x^2}}$

$\dot{y} = 0 = b - ay - x^2 y \rightarrow \boxed{y_2 = \dfrac{b}{a+x^2}}$

b) Fixed Point(s): Set $\boxed{y_1 = y_2} \rightarrow \dfrac{x}{\cancel{a+x^2}} = \dfrac{b}{\cancel{a+x^2}} \rightarrow x = b$

$$\boxed{x^* = \left(b, \frac{b}{a+b^2}\right)}$$

Sketch for: $\boxed{a = \frac{1}{2} \quad b = 1}$



c) How to get arrow <u>directions</u> on nullclines?

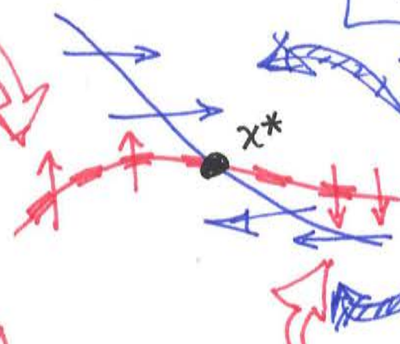
Zoom in near x^* (where $\dot{x} = \dot{y} = 0$).

$$\boxed{\dot{x} = -x + (a + x^2)y}$$

$\dot{x} = 0$ @ x^*

$\dot{y} = b - ay - x^2 y$

$\dot{y} = 0$ @ x^*



$\boxed{\dot{x} > 0 \text{ as } y\uparrow}$ (wrt x^*)

$\boxed{\dot{x} < 0 \text{ as } y\downarrow}$

$\dot{y} = (b - ay) - x^2 y$

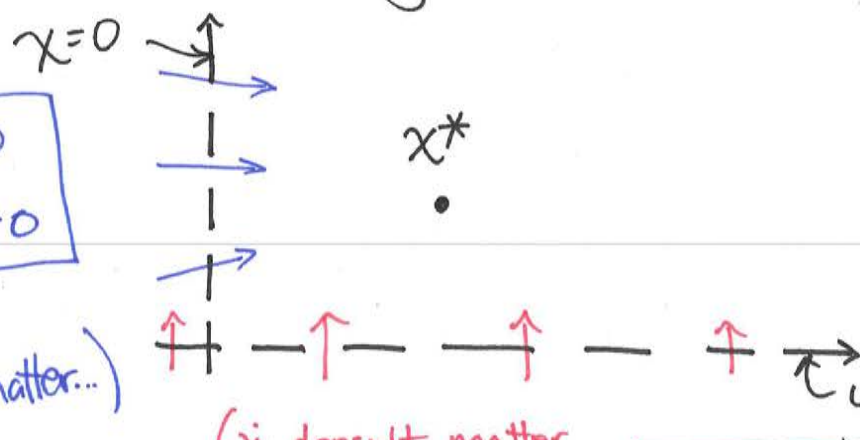
$\boxed{\dot{y} < 0 \text{ as } x^2\uparrow}$ (wrt x^*)

$\boxed{\dot{y} > 0 \text{ as } x^2\downarrow}$ (local to x^*)

$\therefore$ Clockwise flow about x^*

d) How to get a <u>trapping</u> region?

- Look for lines (or curves) where flow goes strictly "one way" along entire boundary of interest:



$\boxed{@ x=0 \quad \dot{x} = ay > 0}$

(Here, $\dot{y}$ doesn't matter...)

($\dot{x}$ doesn't matter along $y=0$. just $\dot{y}$...)

$\boxed{@ y=0, \; \dot{y} = b > 0}$

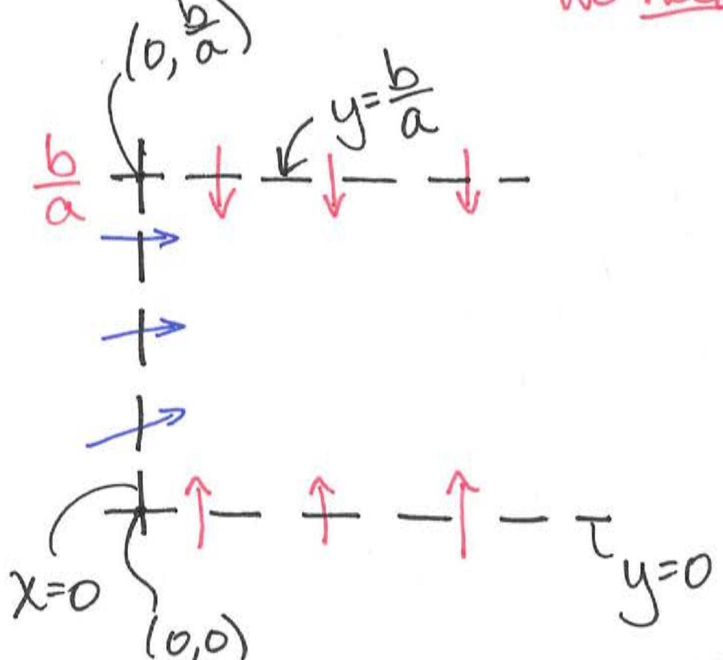
- So far, we have 2 sides.

→ Where can we draw a "roof", such that $\underline{\dot{y}<0}$ is guaranteed (if anywhere)?

$\dot{y} = b - ay - x^2 y$ ← for $x=0$ (y-axis),

we $\underline{\text{need}}$ $\dot{y} = b - ay < 0$

$$\boxed{y > \frac{b}{a}}$$



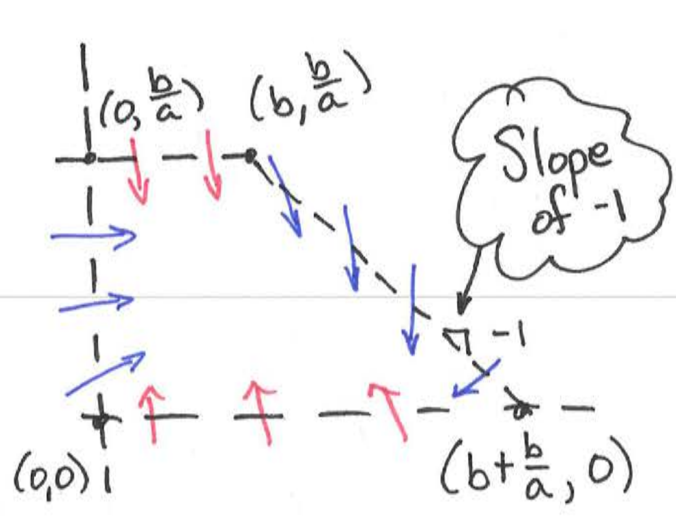
- One side left! Note terms cancel out for $\dot{x} + \dot{y}$:

$$\dot{x} = -x + (a + x^2) y$$
$$\dot{y} = b - (a + x^2) y$$
$$\overline{\dot{x} + \dot{y} = b - x}$$

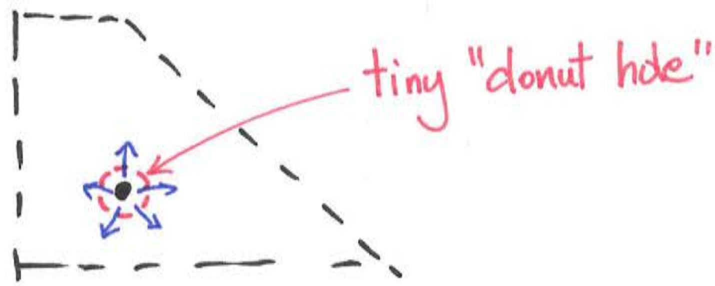
↓

$\therefore \dot{y} = -\dot{x}$ @ $x = b$

$$\boxed{\boxed{\dot{y} < -\dot{x} \text{ for } x > b}}$$



• Finally we must excluded the fixed pt AND guarantee flow @ this inner bound goes into the TRAP.



→ x^* needs to be an unstable node.

Linearization:

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \left. \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{bmatrix} \right|_{x^*}$$

$x^* = \left(b, \frac{b}{a+b^2}\right)$, so

$$A = \begin{bmatrix} -1+2\frac{b^2}{a+b^2} & (a+b^2) \\ -2\frac{b^2}{a+b^2} & -(a+b^2) \end{bmatrix}$$

$$\Delta = a+b^2 > 0$$

$$\tau = \frac{-b^2+(2a-1)b^2+(a+a^2)}{a+b^2}$$

x^* unstable REQUIRES a in the area shown, bounded by

$$b^2 = \tfrac{1}{2}\left(1-2a \pm \sqrt{1-8a}\right)$$