Core Matrix Analysis

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Contents

1	A Note to the Student	3
1.1	Acknowledgements	5
2	Matrix Arithmetic	6
2.1	Notation	6
2.2	Addition & Subtraction	7
2.3	Multiplication	8
2.4	Inverses	10
2.5	Transpose	12
2.6	Gaussian Elimination	13
2.7	Solving $\mathbf{Ax} = \mathbf{b}$	17
2.8	Problems	19
3	Geometry	20
3.1	Vector Spaces	20
3.2	Hyper-planes	22
3.3	Lengths	25
3.4	Angles	28
3.5	Matrix Norms	30
3.6	Riesz–Thorin	33
3.7	Perturbed inverses	37
4	Orthogonality	39
4.1	Unitary Matrices	39
4.2	The Singular Value Decomposition	40
4.3	Orthogonal Subspaces	43
4.4	Minimum norm least-squares solution	45
4.5	Problems	48
5	Spectral Theory	49
5.1	Spectral Decompositions	49
5.2	Invariant subspaces	57
5.3	Difference Equations	60
5.4	Matrix-valued functions	62
5.5	Functions of matrices	64
5.6	Differential equations	67
5.7	Localization of eigenvalues	69
5.8	Real symmetric matrices	71
5.9	Cholesky factorization	76

5.10 Problems 77

6 Tensor Algebra 78
6.1 Kronecker product 78
6.2 Tensor Product Spaces 80
6.3 Symmetric tensors 83
6.4 Symmetric tensor powers 94
6.5 Signs of permutations 96
6.6 Anti-symmetric tensors 98

1 A Note to the Student

These notes are very much a work in progress. Please check the web-site frequently for updates.

These notes do not attempt to explain matrix analysis or even linear algebra. For that I recommend other texts. For example the chapter on *Matrix Arithmetic* is more of an extended exercise than an explanation.

If you need explanations then G. Strang's *Linear Algebra and its Applications*, is a very good introduction for the neophyte.

On the other hand, if you have had a prior introduction to linear algebra, then C. Meyer's *Matrix Analysis and Applied Linear Algebra* is an excellent choice.

For students interested in systems theory, controls theory or operator theory I recommend H. Dym's *Linear Algebra in Action*.

Finally, for students of mathematics, I suggest *A (Terse) Introduction to Linear Algebra* by Y. Katznelson and Y. R. Katznelson.

After this class to see how the ideas presented here can be generalized to the infinite-dimensional setting I recommend I. Gohberg, M. Kaashoek and S. Goldberg's *Basic Classes of Linear Operators*. Another excellent book is P. Lax's *Functional Analysis*.

For more results in matrix analysis with good explanations nothing can beat R. Horn and C. Johnson's classic *Matrix Analysis*.

The serious student of mathematics will also want to look at R. Bhatia's *Matrix Analysis*.

For all algorithmic issues *Matrix Computations* by G. H. Golub and C. van Loan is a classic source.

I hope these notes relieve the student of the burden of taking handwritten notes in my lectures. Anyway, a good way to learn the subject is to go through these notes working out all the exercises.

These notes are still a work in progress—typos abound. Please email them me as you find them (**email**: shiv@ece.ucsb.edu).

Ideally there should be no errors in the proofs. If there are I would appreciate hearing about them.

There are many ways to present matrix analysis. My desire has been to find short, constructive approaches to all proofs. If you have a shorter and more constructive proof for any of the material please let me know.

Almost all proofs presented here are well-known. If at all there is a claim to innovation it *might* be in the proof of the Jordan decomposition theorem. What is uncommon is a presentation of a version of the Riesz–Thorin interpolation theorem, and a related result of Holmgren. The latter especially is a very useful result that is not as well-known as it should be. Both of these are based on the more general presentation in Lax's *Functional Analysis* book.

The last (incomplete) chapter on tensor algebra is very much a work in progress and could easily stand a couple of re-writes. Use with a great deal of caution.

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If I have inadvertently left somebody out please let me know.

2 Matrix Arithmetic

2.1 Notation

$\circ$ $g \circ f$ denotes the composition of the function g with the function f; that is, $(g \circ f)(x) = g(f(x))$. **1**

$\mathbb{N}$ The set of all positive integers. **2**

$\mathbb{Z}$ The set of all integers. **3**

$\mathbb{R}$ The set of all real numbers. **4**

$\mathbb{C}$ The set of all complex numbers. **5**

Scalar For us scalars will denote either real numbers or complex numbers. The context will make it clear which one we are talking about. Small Greek letters $\alpha, \beta, \gamma, \ldots$ will usually denote scalars. **6**

Matrix A matrix is a rectangular array of scalars. If $\mathbf{A}$ is a matrix then the scalar at the intersection of row i and column j is denoted by $A_{i,j}$. **7**

$m \times n$ An $m \times n$ matrix has m rows and n columns. One, or both, of m and n can be zero. **8**

$\mathbb{R}^{m \times n}$ The set of all real $m \times n$ matrices. **9**

$\mathbb{C}^{m \times n}$ The set of all complex $m \times n$ matrices. **10**

$\mathbb{R}^n$ $\mathbb{R}^{n \times 1}$, also called the set of column vectors with n real components. **11**

$\mathbb{C}^n$ $\mathbb{C}^{n \times 1}$, also called the set of column vectors with n complex components. **12**

Block Matrix A block matrix is a rectangular array of matrices. If $\mathbf{A}$ is a block matrix then the matrix at the intersection of block row i and block column j is denoted by $\mathbf{A}_{i,j}$. We will assume that all matrices in block column j have n_j columns, and all matrices in block row i will have m_i rows. That is we will assume that $\mathbf{A}_{i,j}$ is an $m_i \times n_j$ matrix. We will denote the block matrix $\mathbf{A}$ pictorially as follows **13**

$$\mathbf{A} = \begin{array}{c} \\ m_1 \\ \vdots \\ m_k \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & \cdots & n_l \end{array} \\ \begin{pmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,l} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{1,k} & \cdots & \mathbf{A}_{k,l} \end{pmatrix} \end{array}.$$

This is also called a $k \times l$ **block partitioning** of the matrix $\mathbf{A}$.

2.2 Addition & Subtraction

Scalar multiplication For any scalar α **14**

$$
\begin{array}{lcl}
\alpha \mathbf{A}_{m\times n} & = & \mathbf{B}_{m\times n} \\
\alpha\,(\,a\,)_{1\times 1} & = & (\,\alpha a\,)_{1\times 1} \\
\alpha \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix} & = & \begin{pmatrix} \alpha\mathbf{A}_{1,1} & \alpha\mathbf{A}_{1,2} \\ \alpha\mathbf{A}_{2,1} & \alpha\mathbf{A}_{2,2} \end{pmatrix}
\end{array}
$$

The above definition of scalar multiplication must be interpreted as follows. The first equation implies that the argument $\mathbf{A}$ and the result $\mathbf{B}$ must have identical number of rows m, and columns n. Therefore if either m or n is zero there are no entries in $\mathbf{B}$ and nothing to compute. If the argument is a 1×1 matrix the second equation states how the result must be computed. If the argument is larger than that, the third equation states how the scalar multiplication can be reduced into at most four smaller scalar multiplications.

Exercise 1 Prove that if $\alpha\mathbf{A} = \mathbf{B}$ then $\alpha A_{i,j} = B_{i,j}$.

Addition **15**

$$
\begin{array}{lclcl}
\mathbf{A}_{m\times n} & + & \mathbf{B}_{m\times n} & = & \mathbf{C}_{m\times n} \\
(\,a\,)_{1\times 1} & + & (\,b\,)_{1\times 1} & = & (\,a+b\,)_{1\times 1} \\
\begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix} & + & \begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{pmatrix} & = & \begin{pmatrix} \mathbf{A}_{1,1}+\mathbf{B}_{1,1} & \mathbf{A}_{1,2}+\mathbf{B}_{1,2} \\ \mathbf{A}_{2,1}+\mathbf{B}_{2,1} & \mathbf{A}_{2,2}+\mathbf{B}_{2,2} \end{pmatrix}
\end{array}
$$

Exercise 2 Prove that if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, then $A_{i,j} + B_{i,j} = C_{i,j}$.

Subtraction $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$. **16**

Exercise 3 Prove that if $\mathbf{A} - \mathbf{B} = \mathbf{C}$, then $A_{i,j} - B_{i,j} = C_{i,j}$.

0 We denote the $m \times n$ matrix of zeros by $\mathbf{0}_{m\times n}$. We will drop the subscripts if the size is obvious from the context. **17**

Exercise 4 Show that $\mathbf{A} + \mathbf{0} = \mathbf{A}$ and $0\mathbf{A} = \mathbf{0}$.

2.3 Multiplication

$$\begin{array}{ccc}
\mathbf{A}_{m\times k}\mathbf{B}_{k\times n} & = & \mathbf{C}_{m\times n} \\
(\ )_{1\times 0}(\ )_{0\times 1} & = & (\,0\,)_{1\times 1} \\
(\,a\,)_{1\times 1}(\,b\,)_{1\times 1} & = & (\,ab\,)_{1\times 1} \\
\begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}\begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{pmatrix} & = & \begin{pmatrix} \mathbf{A}_{1,1}\mathbf{B}_{1,1}+\mathbf{A}_{1,2}\mathbf{B}_{2,1} & \mathbf{A}_{1,1}\mathbf{B}_{1,2}+\mathbf{A}_{1,2}\mathbf{B}_{2,2} \\ \mathbf{A}_{2,1}\mathbf{B}_{1,1}+\mathbf{A}_{2,2}\mathbf{B}_{2,1} & \mathbf{A}_{2,1}\mathbf{B}_{1,2}+\mathbf{A}_{2,2}\mathbf{B}_{2,2} \end{pmatrix}
\end{array}$$

Exercise 5 Show that if $\mathbf{AB} = \mathbf{C}$ then $\sum_k A_{i,k}B_{k,j} = C_{i,j}$.

Exercise 6 Show that $\mathbf{A}\begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} & \cdots & \mathbf{B}_{1,n} \end{pmatrix} = \begin{pmatrix} \mathbf{AB}_{1,1} & \mathbf{AB}_{1,2} & \cdots & \mathbf{AB}_{1,n} \end{pmatrix}$. This shows that matrix multiplication from the left acts on each (block) column of the right matrix independently.

Exercise 7 Show that

$$\begin{pmatrix} \mathbf{A}_{1,1} \\ \mathbf{A}_{2,1} \\ \vdots \\ \mathbf{A}_{m,1} \end{pmatrix}\mathbf{B} = \begin{pmatrix} \mathbf{A}_{1,1}\mathbf{B} \\ \mathbf{A}_{2,1}\mathbf{B} \\ \vdots \\ \mathbf{A}_{m,1}\mathbf{B} \end{pmatrix}$$

This shows that matrix multiplication from the left acts on each (block) row of the left matrix independently.

Exercise 8 Show that

$$\begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,k} \end{pmatrix}\begin{pmatrix} \mathbf{B}_{1,1} \\ \mathbf{B}_{2,1} \\ \vdots \\ \mathbf{B}_{k,1} \end{pmatrix} = \sum_{l=1}^{k} \mathbf{A}_{1,l}\mathbf{B}_{l,1}$$

This is called a (block) **inner product**. Quite confusingly, when all the partitions have only one row or column, each term on the right in the sum is an outer product. In that case this formula is called the outer product form of matrix multiplication. Usually the term inner product is reserved for the case when $\mathbf{A}$ has one row and $\mathbf{B}$ has one column.

Exercise 9 Show that

$$\begin{pmatrix} \mathbf{A}_{1,1} \\ \mathbf{A}_{2,1} \\ \vdots \\ \mathbf{A}_{m,1} \end{pmatrix}\begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} & \cdots & \mathbf{B}_{1,n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{1,1}\mathbf{B}_{1,1} & \mathbf{A}_{1,1}\mathbf{B}_{1,2} & \cdots & \mathbf{A}_{1,1}\mathbf{B}_{1,n} \\ \mathbf{A}_{2,1}\mathbf{B}_{1,1} & \mathbf{A}_{2,1}\mathbf{B}_{1,2} & \cdots & \mathbf{A}_{2,1}\mathbf{B}_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{A}_{m,1}\mathbf{B}_{1,1} & \mathbf{A}_{m,1}\mathbf{B}_{1,2} & \cdots & \mathbf{A}_{m,1}\mathbf{B}_{1,n} \end{pmatrix}$$

This is called a (block) **outer product**. Usually the term outer produce is reserved for the case when **A** has one column and **B** has one row.

Lower triangular matrix A square matrix **L** is said to be lower triangular if all its entries above the diagonal are zero; that is, $L_{i,j} = 0$ for $i < j$. **19**

Exercise 10 Show that the product of lower triangular matrices is lower triangular.

2.4 Inverses

Left inverse Let A and B be two sets. A function $f : A \to B$ is said to have a left inverse **20**
$g : B \to A$ if $g \circ f$ is the identity map on A.

Exercise 11 Show that a function has a left inverse iff it is one-to-one.

Exercise 12 When does a one-to-one function have more than one left inverse?

Right inverse Let A and B be two sets. A function $f : A \to B$ is said to have a right inverse **21**
$g : B \to A$ if $f \circ g$ is the identity map on B.

Exercise 13 Show that a function has a right inverse iff it is onto.

Exercise 14 When does an onto function have more than one right inverse?

Identity The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$ and is defined to have ones on the **22**
diagonal and zeros every where else. That is, $I_{i,i} = 1$ and $I_{i,j} = 0$ if $i \neq j$.

Exercise 15 Show that $\mathbf{I}_m \mathbf{A}_{m \times n} = \mathbf{A}_{m \times n} \mathbf{I}_n = \mathbf{A}_{m \times n}$.

We will restrict our attention to *linear* left and right inverses of matrices. So we re-define these notions to suit our usage.

Left inverse $\mathbf{A}^{-L}$ is said to be a left inverse of $\mathbf{A}$ if $\mathbf{A}^{-L}\mathbf{A} = \mathbf{I}$. **23**

From now on the subscript on the identity matrix that denotes its size will be dropped if it can be inferred from the context. So, in the above definition, it is clear that the size of the identity matrix is determined by the number of columns of the matrix $\mathbf{A}$.

Exercise 16 How many rows and columns must $\mathbf{A}^{-L}$ have?

Right inverse $\mathbf{A}^{-R}$ is said to be a right inverse of $\mathbf{A}$ if $\mathbf{A}\mathbf{A}^{-R} = \mathbf{I}$. **24**

Exercise 17 How many rows and columns must $\mathbf{A}^{-R}$ have?

To unify our definition of matrix inverses with function inverses we can think of a matrix $\mathbf{A}_{m \times n}$ as a function that maps vectors in $\mathbb{C}^n$ to vectors in $\mathbb{C}^m$ by the rule $y = Ax$ for all $x \in \mathbb{C}^n$.

Exercise 18 Verify that the above statement makes sense; that is, if $\mathbf{A}^{-L}$ is a matrix left inverse for $\mathbf{A}_{m \times n}$, then it is also a left inverse for $\mathbf{A}$ viewed as a function from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Inverse $\mathbf{A}^{-1}$ is said to be an inverse of $\mathbf{A}$ if it is both a left and right inverse of $\mathbf{A}$. **25**

Exercise 19 Show that if $\mathbf{A}^{-1}$ exists then it must be unique. *Hint*: Use Exercise 11, Exercise 12, Exercise 13, Exercise 14 and Exercise 18.

Example 1

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

when $ad - bc \neq 0$.

Example 2

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1} \end{pmatrix}$$

when $\mathbf{A}^{-1}$ and $\mathbf{C}^{-1}$ exist.

Exercise 20 Find

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}^{-1}$$

when $\mathbf{A}^{-1}$ and $\mathbf{C}^{-1}$ exist.

Example 3

$$\begin{pmatrix} \mathbf{I} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{X} \end{pmatrix} = \mathbf{I}$$

but

$$\begin{pmatrix} \mathbf{I} \\ \mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}.$$

This shows that a left inverse need not be a right inverse and vice versa.

Exercise 21 Show that the matrix

$$\begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}$$

has no left or right inverses. Later we will define the *pseudo-inverse* of a matrix, which will always exist.

Upper triangular matrix A square matrix $\mathbf{U}$ is said to be upper triangular if all its entries below the diagonal are zero; that is, $U_{i,j} = 0$ for $i > j$. **26**

Exercise 22 Show that the inverse of an upper triangular matrix exists if all the diagonal entries are non-zero, and that the inverse is also upper triangular. *Hint*: Use Exercise 20.

Exercise 23 Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ when $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist.

2.5 Transpose

Transpose Transpose is denoted by a raised superscript T and is defined by **27**

$$\begin{aligned} (\,a\,)_{1\times 1}^T &= (\,a\,)_{1\times 1} \\ \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}^T &= \begin{pmatrix} \mathbf{A}_{1,1}^T & \mathbf{A}_{2,1}^T \\ \mathbf{A}_{1,2}^T & \mathbf{A}_{2,2}^T \end{pmatrix} \end{aligned}$$

Exercise 24 Show that if $\mathbf{B}_{n\times m} = \mathbf{A}^T$ then $\mathbf{A}$ is an $m \times n$ matrix and $B_{i,j} = A_{j,i}$.

Exercise 25 Show that $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

Exercise 26 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ provided the product $\mathbf{AB}$ is well-defined.

Hermitian transpose Hermitian transpose is denoted by a raised superscript H and is defined by **28**

$$\begin{aligned} (\,a\,)_{1\times 1}^H &= (\,\bar{a}\,)_{1\times 1} \\ \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}^H &= \begin{pmatrix} \mathbf{A}_{1,1}^H & \mathbf{A}_{2,1}^H \\ \mathbf{A}_{1,2}^H & \mathbf{A}_{2,2}^H \end{pmatrix} \end{aligned}$$

where $\bar{z}$ denotes the complex conjugate of z.

Exercise 27 Show that if $\mathbf{B}_{n\times m} = \mathbf{A}^H$ then $\mathbf{A}$ is an $m \times n$ matrix and $B_{i,j} = \bar{A}_{j,i}$.

Exercise 28 Show that $(\mathbf{A}+\mathbf{B})^H = \mathbf{A}^H + \mathbf{B}^H$.

Exercise 29 Show that $(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H$ provided the product $\mathbf{AB}$ is well-defined.

The (Hermitian) transpose is a crucial operator as it lets $m \times n$ matrices act by matrix multiplication on other $m \times n$ matrices.

Exercise 30 Show that $\mathbf{A}^H\mathbf{A}$ and $\mathbf{AA}^H$ are well-defined matrix products. Note that in general $\mathbf{A}^2$ is *not* a well-defined matrix product.

2.6 Gaussian Elimination

How do we compute a left, right or just plain old inverse of a given matrix $\mathbf{A}$? Answer: by Gaussian elimination. We will present Gaussian elimination as a matrix factorization.

Permutation Given a permutation $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the integers $1, \ldots, n$ we can define a permutation matrix $\mathbf{P}$ by the equation **29**

$$\mathbf{P}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma_1} \\ x_{\sigma_2} \\ \vdots \\ x_{\sigma_n} \end{pmatrix} \quad x_i \in \mathbb{C}.$$

Exercise 31 Write $\mathbf{P}$ down explicitly when $\sigma_1 = 4, \sigma_2 = 1, \sigma_3 = 2, \sigma_4 = 3$.

Exercise 32 Write $\mathbf{P}$ down explicitly in the general case.

Exercise 33 Show that $\mathbf{P}^T = \mathbf{P}^{-1}$.

Exercise 34 Show that a product of permutation matrices is another permutation matrix.

Exercise 35 If $\mathbf{P}$ is a permutation matrix such that

$$\mathbf{P}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma_1} \\ x_{\sigma_2} \\ \vdots \\ x_{\sigma_n} \end{pmatrix} \quad x_i \in \mathbb{C}.$$

for some permutation σ_i of the integers $1, \ldots, n$ find

$$\begin{pmatrix} x_1 & x_2 & \cdots x_n \end{pmatrix}\mathbf{P}.$$

Hint: Transpose.

Unit lower triangular matrix A lower triangular matrix with ones on the main diagonal is called a unit lower triangular matrix. **30**

Exercise 36 Show that the product of unit lower triangular matrices is unit lower triangular.

Exercise 37 Show that a unit lower triangular matrix always has an inverse, which is also unit lower triangular, *Hint*: Use Example 2 and Exercise 22.

LU For every $m \times n$ matrix $\mathbf{A}$ there exists two permutations $\mathbf{P}_1$ and $\mathbf{P}_2$ such that $\mathbf{P}_1\mathbf{A}\mathbf{P}_2 = \mathbf{L}\mathbf{U}$, where $\mathbf{L}$ is a unit lower triangular matrix and $\mathbf{U}$ is of the form **31**

$$\mathbf{U} = \begin{array}{c} \\ r \\ m-r \end{array}\begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & 0 \end{pmatrix} \end{array},$$

where $U_{1,1}$ is an upper triangular matrix with non-zero diagonal entries.

Rank The integer r in the **LU** factorization of $\mathbf{A}$ is called the **rank** of the matrix $\mathbf{A}$. **32**

Exercise 38 Give examples of $m \times n$ matrices for which the ranks are 0, 1, m and n.

Proof of LU decomposition. The proof is by induction on the matrix size.

Case 1.

$$\underbrace{\mathbf{I}}_{\mathbf{P}_1}\underbrace{\mathbf{0}}_{\mathbf{A}}\underbrace{\mathbf{I}}_{\mathbf{P}_2} = \underbrace{\mathbf{I}}_{\mathbf{L}}\underbrace{\mathbf{0}}_{\mathbf{U}}$$

In this case $\mathbf{U}_{1,1}$ is empty and the rank $r = 0$.

Case 2.

$$\underbrace{\mathbf{I}}_{\mathbf{P}_1}\underbrace{(\,a\,)_{1\times 1}}_{\mathbf{A}}\underbrace{\mathbf{I}}_{\mathbf{P}_2} = \underbrace{\mathbf{I}}_{\mathbf{L}}\underbrace{(\,a\,)_{1\times 1}}_{\mathbf{U}}$$

If $a \neq 0$ then $r = 1$, otherwise $r = 0$ and $\mathbf{U}_{1,1}$ is empty.

Case 3. Pick two intermediate permutations $\mathbf{Q}_1$ and $\mathbf{Q}_2$ such that the $(1, 1)$ entry of $\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2$ is non-zero.

Exercise 39 Prove that this step is possible if $\mathbf{A} \neq \mathbf{0}$. Otherwise we are done by case 1.

Let

$$\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2 = \begin{array}{c} \\ 1 \\ m-1 \end{array}\begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{pmatrix} A_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix} \end{array}$$

with $A_{1,1} \neq 0$. Let

$$\mathbf{L}_1 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{A}_{2,1}A_{1,1}^{-1} & \mathbf{I} \end{pmatrix} \quad \text{and} \quad \mathbf{U}_1 = \begin{pmatrix} A_{1,1} & \mathbf{A}_{1,2} \\ 0 & \mathbf{A}_{2,2} - \mathbf{A}_{2,1}A_{1,1}^{-1}\mathbf{A}_{1,2} \end{pmatrix}$$

where $\mathbf{L}_1$ is a unit lower triangular matrix. $\mathbf{L}_1$ is called an **elementary Gauss transform**.

Exercise 40 Show that

$$\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2 = \mathbf{L}_1\mathbf{U}_1.$$

Let $\mathbf{S}_1 = \mathbf{A}_{2,2} - \mathbf{A}_{2,1}A_{1,1}^{-1}\mathbf{A}_{1,2}$, which is called a **Schur complement**. Note that $\mathbf{S}_1$ is smaller than $\mathbf{A}$. If $\mathbf{S}_1$ is empty then we are done. Otherwise, by the induction hypothesis $\mathbf{S}_1$ has an **LU** decomposition

$$\mathbf{Q}_3\mathbf{S}_1\mathbf{Q}_4 = \mathbf{L}_2\mathbf{U}_2 \tag{2.1}$$

where $\mathbf{Q}_3$ and $\mathbf{Q}_4$ are the associated permutation matrices. Substituting this in the expression for $\mathbf{U}_1$ we obtain

$$\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2 = \underbrace{\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{A}_{2,1}A_{1,1}^{-1} & \mathbf{I} \end{pmatrix}}_{\mathbf{L}_1} \underbrace{\begin{pmatrix} A_{1,1} & \mathbf{A}_{1,2} \\ 0 & \mathbf{Q}_3^T\mathbf{L}_2\mathbf{U}_2\mathbf{Q}_4^T \end{pmatrix}}_{\mathbf{U}_1}.$$

Exercise 41 Verify this. *Hint*: Multiply equation 2.1 from the left by $\mathbf{Q}_3^T$.

We can now expand and factor the right hand side of the above expression to obtain

$$\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_3^T \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{Q}_3\mathbf{A}_{2,1}A_{1,1}^{-1} & \mathbf{L}_2 \end{pmatrix} \begin{pmatrix} A_{1,1} & \mathbf{A}_{1,2}\mathbf{Q}_4 \\ 0 & \mathbf{U}_2 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_4^T \end{pmatrix}.$$

Exercise 42 Verify this.

We observe that

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_3^T \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_4^T \end{pmatrix}$$

are permutation matrices.

Exercise 43 Prove it.

Therefore their inverses are just their transposes. We can multiply by their transposes on the left and right respectively of the above equation and obtain the desired **LU** decomposition of $\mathbf{A}$

$$\underbrace{\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_3 \end{pmatrix} \mathbf{Q}_1}_{\mathbf{P}_1} \mathbf{A} \underbrace{\mathbf{Q}_2 \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_4 \end{pmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{Q}_3\mathbf{A}_{2,1}A_{1,1}^{-1} & \mathbf{L}_2 \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} A_{1,1} & \mathbf{A}_{1,2}\mathbf{Q}_4 \\ 0 & \mathbf{U}_2 \end{pmatrix}}_{\mathbf{U}}.$$

□

Exercise 44 Verify that $\mathbf{L}$ in the above equation is unit lower triangular and that $\mathbf{U}$ has the form promised in the **LU** decomposition definition 31.

Exercise 45 Write a software program in your favorite programming language to compute the **LU** decomposition of a matrix.

Gaussian elimination, and hence the **LU** decomposition, is the heart of matrix algebra. Schur complements are one common manifestation which often goes completely unnoticed in practice.

2.7 Solving $\mathbf{Ax} = \mathbf{b}$

Given an $m \times n$ matrix $\mathbf{A}$ and an $m \times k$ matrix $\mathbf{b}$ how do we find all $n \times k$ matrices $\mathbf{x}$ which satisfy the equation $\mathbf{Ax} = \mathbf{b}$? Answer: $\mathbf{LU}$ decomposition.

Let $\mathbf{P}_1\mathbf{AP}_2 = \mathbf{LU}$. Substituting this in the equation for $\mathbf{x}$ we obtain the following set of equivalent equations for $\mathbf{x}$

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{P}_1^T\mathbf{LUP}_2^T\mathbf{x} &= \mathbf{b} \\ \mathbf{UP}_2^T\mathbf{x} &= \mathbf{L}^{-1}\mathbf{P}_1\mathbf{b}. \end{aligned}$$

Exercise 46 Why do each of the above equations determine exactly the same set of solutions $\mathbf{x}$?

Let

$$\mathbf{U} = \begin{array}{c} \\ r \\ m-r \end{array} \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & 0 \end{pmatrix} \end{array}$$

where r is a rank of $\mathbf{A}$ and let

$$\mathbf{P}_2^T\mathbf{x} = \mathbf{y} = \begin{array}{c} r \\ n-r \end{array} \begin{pmatrix} \mathbf{y}_{1,1} \\ \mathbf{y}_{2,1} \end{pmatrix} \quad \text{and} \quad \mathbf{L}^{-1}\mathbf{P}_1\mathbf{b} = \begin{array}{c} r \\ m-r \end{array} \begin{pmatrix} \mathbf{b}_{1,1} \\ \mathbf{b}_{2,1} \end{pmatrix},$$

with some abuse of notation. Substituting back into the equation for $\mathbf{x}$ we obtain

$$\begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_{1,1} \\ \mathbf{y}_{2,1} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{1,1} \\ \mathbf{b}_{2,1} \end{pmatrix}.$$

We see that the last block equation *requires* that $\mathbf{b}_{2,1} = \mathbf{0}$. Either this matrix has zero rows and the condition is trivially satisfied, or it does not, and then the validity of this equation depends entirely on the given $\mathbf{b}$ and $\mathbf{L}$ and $\mathbf{P}_1$. If $\mathbf{b}_{2,1} \neq \mathbf{0}$ then there are *no* matrices $\mathbf{x}$ which satisfy the equation $\mathbf{Ax} = \mathbf{b}$. If $\mathbf{b}_{2,1} = \mathbf{0}$ then we must look at the remaining first block equation $\mathbf{U}_{1,1}\mathbf{y}_{1,1} + \mathbf{U}_{1,2}\mathbf{y}_{2,1} = \mathbf{b}_{1,1}$. Since we are guaranteed that $\mathbf{U}_{1,1}$ is invertible we see that the general solution is $\mathbf{y}_{1,1} = \mathbf{U}_{1,1}^{-1}(\mathbf{b}_{1,1} - \mathbf{U}_{1,2}\mathbf{y}_{2,1})$, where we are free to pick $\mathbf{y}_{2,1}$ freely.

Exercise 47 Verify this last statement thoroughly; that is, show that *any* solution $\mathbf{y}$, can be written in this form.

We can state this result more succinctly as

$$\mathbf{y} = \begin{pmatrix} \mathbf{U}_{1,1}^{-1}\mathbf{b}_{1,1} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\mathbf{U}_{1,1}^{-1}\mathbf{U}_{1,2} \\ \mathbf{I} \end{pmatrix} \mathbf{z},$$

where $\mathbf{z}$ can be chosen freely.

Of course we really want all the solutions $\mathbf{x}$ which we now obtain as

$$\mathbf{x} = \mathbf{P}_2 \begin{pmatrix} \mathbf{U}_{1,1}^{-1}\mathbf{b}_{1,1} \\ \mathbf{0} \end{pmatrix} + \mathbf{P}_2 \begin{pmatrix} -\mathbf{U}_{1,1}^{-1}\mathbf{U}_{1,2} \\ \mathbf{I} \end{pmatrix} \mathbf{z}, .$$

whenever $\mathbf{b}_{2,1} = \mathbf{0}$; otherwise there are no solutions.

Exercise 48 Verify that every solution is of this form.

Exercise 49 Show that an $m \times n$ matrix $\mathbf{A}$ has a right inverse iff $\mathbf{rank}(\mathbf{A}) = m$. Such a matrix is called a **full row-rank** matrix. Write down explicitly all right inverses of $\mathbf{A}$. *Hint*: I just did it.

Exercise 50 Find all $\mathbf{x}$ that satisfy the equation $\mathbf{x}^H\mathbf{A} = \mathbf{b}^H$ explicitly in terms of the **LU** factorization of $\mathbf{A}$ (not $\mathbf{A}^H$).

Exercise 51 Show that an $m \times n$ matrix $\mathbf{A}$ has a left inverse iff $\mathbf{rank}(\mathbf{A}) = n$. Such a matrix is called a **full column-rank** matrix. Write down explicitly all left inverses of $\mathbf{A}$ in terms of the **LU** decomposition of $\mathbf{A}$ (not $\mathbf{A}^H$).

Exercise 52 Show that if a matrix has both a left and right inverse then it is square.

Exercise 53 Show that $\mathbf{A}$ has a left inverse iff $\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = 0$.

Exercise 54 Show that $\mathbf{A}$ has a right inverse iff $\mathbf{x}^H\mathbf{A} = \mathbf{0}$ implies $\mathbf{x} = 0$.

2.8 Problems

Problem 1 Find all non-zero solutions $\mathbf{x}$ of $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{0}$. Show that there are non-trivial solutions $\mathbf{x} \neq \mathbf{0}$ if $m < n$.

Problem 2 Find all matrices $\mathbf{b}$ such that $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{b}$ has *no* solution $\mathbf{x}$. Show that such matrices $\mathbf{b}$ always exist if $m > n$.

Usually in practice "linear algebra" is needed to analyze linear equations where the coefficient matrix has some special structure. Here are some simple cases.

Problem 3 Find all matrices $\mathbf{X}$ that satisfy the equation $\mathbf{AXB}^T = \mathbf{C}$, in terms of the $\mathbf{LU}$ factorizations of $\mathbf{A}$ and $\mathbf{B}$. State the precise conditions under which there are no solutions.

Problem 4 Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be two upper-triangular matrices. Let $\mathbf{Z}$ be an $m \times n$ matrix. Let $\mathbf{X}$ be an unknown matrix that satisfies the equation

$$\mathbf{U}_1\mathbf{X} + \mathbf{X}\mathbf{U}_2 = \mathbf{Z}.$$

A. Give an algorithm to find $\mathbf{X}$ in $O(mn(m+n))$ flops (floating-point operations).

B. Find conditions on $\mathbf{U}_1$ and $\mathbf{U}_2$ which guarantee the existence of a unique solution $\mathbf{X}$.

C. Give a non-trivial example ($\mathbf{U}_1 \neq \mathbf{0}$, $\mathbf{U}_2 \neq \mathbf{0}$, $\mathbf{X} \neq \mathbf{0}$) where those conditions are not satisfied and

$$\mathbf{U}_1\mathbf{X} + \mathbf{X}\mathbf{U}_2 = \mathbf{0}.$$

3 Geometry

We will now develop the basic notions of Eulcidean geometry in higher-dimensional spaces.

3.1 Vector Spaces

$\mathbb{F}$ We will use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$, and we will call its elements as **scalars**. **33**

Vector space A vector space consists of a set $\mathcal{V}$ of *vectors* and a set $\mathbb{F}$ of *scalars*, an operation $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called *vector addition*, and an operation called *scalar multiplication* from $\mathcal{V} \times \mathbb{F}$ to $\mathcal{V}$, that satisfy the following properties for all $u, v, w \in \mathcal{V}$ and all $\alpha, \beta \in \mathbb{F}$: **34**

1. $u + v = v + u \in \mathcal{V}$ (closed and commutative);
2. $(u + v) + w = u + (v + w)$ (associative);
3. There exists a 0 vector in $\mathcal{V}$ such that $u + 0 = u$ (existence of identity);
4. For each $u \in V$ there exists an element $-u \in \mathcal{V}$ such that $u+(-u) = 0$ (existence of inverse);
5. $u\alpha \in \mathcal{V}$ (scalar multiplication is closed);
6. $(u + v)\alpha = u\alpha + v\alpha$ (distributive);
7. $u(\alpha + \beta) = u\alpha + u\beta$ (distributive);
8. $u(\alpha\beta) = (u\alpha)\beta$ (associative);
9. $u1 = u$ (unit scaling).

Note: We will allow the scalar in scalar multiplication to be written on wither side of the vector it is multiplying. This is possible because both vector addition and scalar multiplication are commutative, associative and distribute over each other.

Exercise 55 Show that the 0 vector in $\mathcal{V}$ is unique.

Exercise 56 Show that for each $v \in \mathcal{V}$ there is exactly one vector w such that $v + w = 0$.

Exercise 57 Show that $0v = 0$ for all $v \in \mathcal{V}$.

Exercise 58 Show that $(-1)v = -v$ for all $v \in \mathcal{V}$.

$\mathbb{F}^n$ The set of column vectors with n elements drawn from $\mathbb{F}$. **35**

Exercise 59 Show that $\mathbb{F}^n$ is a vector space over the scalars $\mathbb{F}$ with the obvious definition of vector addition and scalar multiplication.

$\mathbb{F}^{m \times n}$ The set of $m \times n$ matrices with elements drawn from $\mathbb{F}$. **36**

Exercise 60 Show that $\mathbb{F}^{m \times n}$ is a vector space over the scalars $\mathbb{F}$ with matrix addition as vector addition and the usual scalar multiplication.

Note: When the scalar $\mathbb{F}$ is obvious, we will abuse notation and call $\mathcal{V}$ as the vector space. There is usually no confusion as to the implied vector addition and scalar multiplication operations either.

3.2 Hyper-planes

Subspace A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if $\mathcal{W}$ is a vector space in its own right. **37**

Fortunately, it turns out that $\mathcal{W}$ is a subspace of $\mathcal{V}$ iff it is closed under vector addition and scalar multiplication.

Exercise 61 Prove it.

Nullspace The nullspace of a matrix $\mathbf{A} \in \mathbb{F}^{m \times n}$, denoted by $\mathcal{N}(\mathbf{A})$, is the set of all column vectors $\mathbf{x} \in \mathbb{F}^n$ such that $\mathbf{Ax} = \mathbf{0}$. **38**

Exercise 62 Show that $\mathcal{N}(\mathbf{A})$ is a subspace.

Range space The range space of a matrix $\mathbf{A} \in \mathbb{F}^{m \times n}$, denoted by $\mathcal{R}(\mathbf{A})$, is the set of all vectors $\mathbf{y} \in \mathbb{F}^m$ such that $\mathbf{Ax} = \mathbf{y}$ for some vector $\mathbf{x}$. This is also called the **column space** of $\mathbf{A}$. **39**

Exercise 63 Show that $\mathcal{R}(\mathbf{A})$ is a subspace.

Left nullspace $\mathcal{N}(\mathbf{A}^H)$ is called the left nullspace of $\mathbf{A}$. **40**

Row space $\mathcal{R}(\mathbf{A}^H)$ is called the row space of $\mathbf{A}$. **41**

Exercise 64 Show that the intersection of two subspaces is a subspace.

Exercise 65 Show that the union of two subspaces need not be a subspace.

Sums of sets Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two subsets of the vector space $\mathcal{V}$. $\mathcal{W}_1 + \mathcal{W}_2$ is defined to be the set of all vectors of the form $w_1 + w_2$, where $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$. **42**

Exercise 66 Show that $\mathcal{W}_1 + \mathcal{W}_2$ is a subspace if $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces.

Exercise 67 Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces. Show that $\mathcal{W}_1 + \mathcal{W}_2$ is the smallest subspace that contains $\mathcal{W}_1 \cup \mathcal{W}_2$.

Exercise 68 Show that $\mathcal{R}\left(\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\right) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})$.

Exercise 69 Show that

$$\mathcal{N}\left(\begin{pmatrix}\mathbf{A} \\ \mathbf{B}\end{pmatrix}\right) = \mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{B}).$$

Direct sum If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces with $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$, then $\mathcal{W}_1 + \mathcal{W}_2$ is written as $\mathcal{W}_1 \oplus \mathcal{W}_2$, and it is called the direct sum of $\mathcal{W}_1$ and $\mathcal{W}_2$. **43**

Linear combination If $v_1, v_2, \ldots, v_k$ $(0 < k < \infty)$ are vectors and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars, then the vector $\sum_{i=1}^{k} \alpha_i v_i$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$. **44**

Note that we can write this as

$$\sum_{i=1}^{k} \alpha_i v_i = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

So matrix vector multiplication results in a linear combination of the columns of the matrix. Note that the matrix containing the vectors v_i must be viewed only as a block matrix, since the vectors v_i are abstract at this point. However, from now on we will allow such abstract block matrix notation where convenient.

Span The span of a set of vectors $v_1, v_2, \ldots, v_k$ is defined to be the set of all possible linear combinations of $v_1, v_2, \ldots, v_k$. **45**

Exercise 70 Show that $\textbf{span}\{v_1, v_2, \ldots, v_k\}$ is a subspace.

Exercise 71 Show that $\textbf{span}\{v_1, v_2, \ldots, v_k\}$ is the smallest subspace that contains $v_1, v_2, \ldots, v_k$.

Exercise 72 Show that $\textbf{span}\{v_1, v_2, \ldots, v_k\} = \mathcal{R}(\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix})$.

Spans are a compact means of specifying a subspace. However, they are not the most compact necessarily.

Linear Independence A set of vectors $v_1, v_2, \ldots, v_k$ is said to be linearly independent if the equation **46**

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0,$$

has only the zero solution $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Linear Dependence A set of vectors $v_1, v_2, \ldots, v_k$ is said to be linearly dependent if they are not linearly independent. **47**

Exercise 73 Show that $v_1, v_2, \ldots, v_k$ are linearly independent iff $\mathcal{N}(\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}) = 0$.

Exercise 74 Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{L} \\ \mathbf{X} \end{pmatrix},$$

where $\mathbf{L}$ is a lower-triangular matrix. Show that the columns of $\mathbf{A}$ are linearly independent if the diagonal entries of $\mathbf{L}$ are non-zero.

Basis A set of vectors $v_1, v_2, \ldots, v_k$ is a basis for a subspace $\mathcal{W}$ if $\mathbf{span}\{v_1, v_2, \ldots, v_k\} = \mathcal{W}$ and the vectors $v_1, v_2, \ldots, v_k$ are linearly independent. **48**

Dimension Suppose a subspace $\mathcal{W}$ has a basis with k vectors. Then k is called *the* dimension of $\mathcal{W}$ and denoted by $\mathbf{dim}(\mathcal{W}) = k$. **49**

Implicit in the above definition is that the dimension of a subspace does not depend on the choice of basis. We prove this now. Assume to the contrary that the subspace $\mathcal{W}$ has $v_1, v_2, \ldots, v_k$ as one basis, and $w_1, w_2, \ldots, w_r$ as a second basis with $r < k < \infty$. It follows from the properties of basis that there is an $r \times k$ matrix $\mathbf{X}$ such that

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & \cdots & w_r \end{pmatrix} \mathbf{X}.$$

Since $\mathbf{X}$ is fat, $\mathcal{N}(\mathbf{X}) \neq \{\mathbf{0}\}$.

Exercise 75 Why?

Let $\mathbf{0} \neq \mathbf{z} \in \mathcal{N}(\mathbf{X})$. Then it follows that

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} \mathbf{z} = \begin{pmatrix} w_1 & w_2 & \cdots & w_r \end{pmatrix} \mathbf{X}\mathbf{z} = 0.$$

Hence $v_1, v_2, \ldots, v_k$ are not linearly independent, giving a contradiction.

Exercise 76 Let $\mathbf{A}$ be an $m \times n$ matrix. Find bases for

- $\mathcal{R}(\mathbf{A})$
- $\mathcal{N}(\mathbf{A})$
- $\mathcal{R}(\mathbf{A}^H)$
- $\mathcal{N}(\mathbf{A}^H)$

explicitly using the $\mathbf{LU}$ factorization of $\mathbf{A}$ (only). From this establish that

- $\mathbf{dim}(\mathcal{R}(\mathbf{A})) = \mathbf{dim}(\mathcal{R}(\mathbf{A}^H)) = \mathbf{rank}(\mathbf{A})$
- $\mathbf{dim}(\mathcal{N}(\mathbf{A})) + \mathbf{rank}(\mathbf{A}) = n$.

The last formula is called the **rank-nullity theorem**.

Exercise 77 Show that $\mathbf{dim}(\mathbb{F}^n) = n$.

Exercise 78 Show that $\mathbf{dim}(\mathbb{F}^{m \times n}) = mn$.

Exercise 79 Let $\mathbb{F}^\infty$ denote the set of columns vectors with elements drawn from $\mathbb{F}$ and indexed from $1, 2, \ldots$. Show that $\mathbf{dim}(\mathbb{F}^\infty)$ is not finite.

Exercise 80 Show that for every matrix $\mathbf{A}$ there are two full column-rank matrices $\mathbf{X}$ and $\mathbf{Y}$ with the same rank as $\mathbf{A}$, such that $\mathbf{A} = \mathbf{X}\mathbf{Y}^H$.

3.3 Lengths

Norm A norm, denoted by $\|\cdot\|$, is a function from a vector space $\mathcal{V}$ over $\mathbb{F}$ to $\mathbb{R}$ that satisfies the following properties **50**

- $\|v\| \geq 0$ for all $v \in \mathcal{V}$ (positive semi-definiteness)
- $\|v\| = 0$ iff $v = 0$ (positive definiteness)
- $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{F}$ and all $v \in \mathcal{V}$ (homogeneity)
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in \mathcal{V}$ (triangle inequality)

Exercise 81 Show that $|\|v\| - \|w\|| \leq \|v - w\|$.

Exercise 82 Show that norms are continuous functions on $\mathbb{F}^n$. *Hint*: Let e_i denote a basis for $\mathbb{F}^n$. Then

$$\|v - w\| \leq \sum_{i=1}^{n} |v_i - w_i|\|e_i\| \leq \text{constant} \cdot \max_{1 \leq i \leq n} |v_i - w_i|.$$

Unit Ball The set of vectors with norm ≤ 1 is called the unit ball of that norm. **51**

Unit Sphere The set of vectors with norm 1 is called the unit sphere for that norm. **52**

Convex sets A set of vectors in a vector space $\mathcal{V}$ is said to be convex if for every pair of vectors v and w in the set, and every $0 \leq \lambda \leq 1$, the vector $\lambda v + (1 - \lambda)w$ is also in the set. **53**

Exercise 83 Show that the intersection of two convex sets is convex.

Exercise 84 Show that the sum of two convex sets is convex.

Exercise 85 Show that the unit ball of a norm is a convex set.

Convex function A function f from a vector space to $\mathbb{R}$ is said to be convex if **54**

$$f(\lambda v + (1 - \lambda)w) \leq \lambda f(v) + (1 - \lambda)f(w)$$

for all vectors v and w and $0 \leq \lambda \leq 1$.

Exercise 86 Show that if f is a convex function then $\{v : f(v) \leq \gamma\}$ is a convex set for all γ.

Exercise 87 By considering the function $-e^x$ show that the converse is not true.

Exercise 88 Show that $\|\cdot\|$ is a convex function.

We claim that if $f : \mathcal{V} \to \mathbb{R}$ is a function that satisfies the following conditions

- $f(v) \geq 0$ for all $v \in \mathcal{V}$
- $f(v) = 0$ iff $v = 0$
- $f(\alpha v) = |\alpha| f(v)$ for all $\alpha \in \mathbb{F}$ and all $v \in \mathcal{V}$
- The set $\{v : f(v) \leq 1\}$ is convex

then f defines a norm on $\mathcal{V}$.

Exercise 89 Show that the ball of radius r, $\{v : f(v) \leq r\}$, is convex.

Exercise 90 Show that $f(\lambda f(x) y + (1-\lambda) f(y) x) \leq f(x) f(y)$ for all $0 \leq \lambda \leq 1$. *Hint*: $f(x)y$ lies in the ball of radius $f(x)f(y)$.

Exercise 91 Finish the proof by picking $\lambda = f(y)/(f(x) + f(y))$ in the above inequality.

This shows that the triangle inequality requirement is equivalent to the convexity of the unit ball.

p-norm For $\mathbf{x} \in \mathbb{F}^n$ the p-norm of $\mathbf{x}$, for $1 \leq p < \infty$ is defined to be **55**

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

For $p = \infty$ we define the ∞-norm of $\mathbf{x}$ to be

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Exercise 92 Show that

$$\lim_{p \uparrow \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty.$$

Exercise 93 Show that the function $\|\cdot\|_p$ for $1 \leq p \leq \infty$ satisfies the first three conditions for being a norm.

Exercise 94 Show that the sum of two convex functions is convex.

Assume that the function $|x|^p$ is convex when $1 \leq p < \infty$. Or, better yet, prove it.

Exercise 95 Show that the function $f_1(\mathbf{x}) = |x_1|^p$ is convex if $1 \leq p < \infty$.

Exercise 96 Show that the function $\|\mathbf{x}\|_p^p$ is convex if $1 \leq p < \infty$.

Exercise 97 Show that the maximum of two convex functions is convex.

Exercise 98 Show that $\|\mathbf{x}\|_\infty$ is convex.

Now observe that the unit ball $\{\mathbf{x} : \|\mathbf{x}\|_p \leq 1\} = \{\mathbf{x} : \|\mathbf{x}\|_p^p \leq 1\}$. It follows that the unit balls for p-norms are convex. Hence, by exercise ??, we have established the triangle inequality for p-norms.

Minkowski's inequality 56

$$\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p, \qquad 1 \leq p \leq \infty.$$

The case $p = 2$ is called the Euclidean norm. Observe that

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^H \mathbf{x}}.$$

Equivalence of norms Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two norms on a vector space $\mathcal{V}$. The two norms are said to be equivalent if there exist two positive finite constants c_1 and c_2 such that 57

$$c_1 \|v\|_\alpha \leq \|v\|_\beta \leq c_2 \|v\|_\alpha, \qquad \forall v \in \mathcal{V}.$$

Theorem 1 All norms on finite dimensional vector spaces are equivalent.

Proof. Since norms are continuous functions it follows that the unit sphere is closed.

Exercise 99 Show that the unit sphere is closed.

Since $\mathcal{V}$ is assumed to be finite dimensional the unit sphere is compact.

Exercise 100 Why?

Therefore the continuous functions $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ must both achieve their minimum and maximum on the unit sphere. From this the existence of the positive finite constants c_1 and c_2 follows. (Why?) □

Exercise 101 Show that for $\mathbf{x} \in \mathbb{F}^n$, $\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q$ for $1 \leq q \leq p \leq \infty$.

Exercise 102 Show that for $\mathbf{x} \in \mathbb{F}^n$, $\|\mathbf{x}\|_2 \leq \sqrt{\|\mathbf{x}\|_1 \|\mathbf{x}\|_\infty}$.

Exercise 103 Establish the following inequalities for $\mathbf{x} \in \mathbb{F}^n$

$$\|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2$$
$$\|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$$
$$\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$$

Hint: For the first inequality use the fact that $2xy \leq |x|^2 + |y|^2$.

3.4 Angles

Pythagorean Theorem: If $\mathbf{x}$ and $\mathbf{y}$ are two perpendicular vectors (whatever that means), they should form a right-angle triangle with $\mathbf{x}+\mathbf{y}$ as the hypotenuse. Then the Pythagorean Theorem would imply that

$$\|\mathbf{x}+\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2.$$

Simplifying this using the fact that $\|\mathbf{x}\|_2^2 = \mathbf{x}^H\mathbf{x}$, we obtain $\mathbf{x}^H\mathbf{y} = 0$.

Orthogonal Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{F}^n$ are said to be (mutually) orthogonal if $\mathbf{x}^H\mathbf{y} = 0$. This is denoted by $\mathbf{x} \perp \mathbf{y}$. **58**

More generally, for vectors in $\mathbb{R}^n$, we define the angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ via the formula

$$\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}.$$

There are many ways to justify this choice. One supporting fact is the Cauchy–Buniakowsky–Schwartz (CBS) inequality.

CBS inequality **59**

$$\left|\mathbf{x}^H\mathbf{y}\right| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

Exercise 104 Given $\mathbf{x}$ and $\mathbf{y}$ from $\mathbb{F}^n$, find λ_* such that,

$$\|\mathbf{x}+\lambda\mathbf{y}\|_2 \geq \|\mathbf{x}+\lambda_*\mathbf{y}\|_2$$

for all $\lambda \in \mathbb{F}$.

Exercise 105 Starting from

$$\|\mathbf{x}+\lambda_*\mathbf{y}\|_2 \geq 0,$$

derive the CBS inequality.

The CBS inequality is a special case of the Hölder inequality.

Hölder inequality **60**

$$\left|\mathbf{x}^H\mathbf{y}\right| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q, \qquad \frac{1}{p}+\frac{1}{q}=1.$$

Exercise 106 Prove the Hölder inequality when $p=1$ and $q=\infty$.

Proof of Hölder inequality. Note that $-\ln x$ is convex on $(0, \infty)$. Hence, for $x > 0$ and $y > 0$,

$$-\ln(\lambda x + (1-\lambda)y) \leq -\lambda \ln x - (1-\lambda)\ln y.$$

Or, equivalently,

$$\lambda \ln x + (1-\lambda)\ln y \leq \ln(\lambda x + (1-\lambda)y)$$

Exponentiating both sides we obtain

$$x^{\lambda} y^{1-\lambda} \leq \lambda x + (1-\lambda)y. \tag{3.1}$$

Therefore it follows that, with $\lambda = \frac{1}{p}$ and $1 - \lambda = \frac{1}{q}$,

$$\left(\frac{|x_i|^p}{\|\mathbf{x}\|_p^p}\right)^{\frac{1}{p}} \left(\frac{|y_i|^q}{\|\mathbf{y}\|_q^q}\right)^{\frac{1}{q}} \leq \frac{1}{p}\frac{|x_i|^p}{\|\mathbf{x}\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|\mathbf{y}\|_q^q}.$$

Summing both sides from 1 to n the Hölder inequality is derived. □

Exercise 107 Show that for $\mathbf{x} \in \mathbb{F}^n$

$$\|\mathbf{x}\|_2 \leq \sqrt{\|\mathbf{x}\|_p \|\mathbf{x}\|_q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Exercise 108 Show that

$$\|\mathbf{x}\|_p = \sup_{\mathbf{0} \neq \mathbf{y} \in \mathbb{F}^n} \frac{\left|\mathbf{x}^H \mathbf{y}\right|}{\|\mathbf{y}\|_q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

For this reason $\|\cdot\|_p$ and $\|\cdot\|_q$ are called **dual norms** whenever $p + q = pq$. $\|\cdot\|_2$ is the only self-dual norm among the lot and plays a prominent role.

3.5 Matrix Norms

Trace The trace of a square matrix is defined to be the sum of its diagonal elements. **61**

Exercise 109 Show that $\mathbf{trace}(\mathbf{A}+\mathbf{B}) = \mathbf{trace}(\mathbf{A}) + \mathbf{trace}(\mathbf{B})$.

Exercise 110 Show that $\mathbf{trace}(\mathbf{AB}) = \mathbf{trace}(\mathbf{BA})$.

Frobenius norm The Frobenius norm of a matrix $\mathbf{A}$, denoted by $\|\mathbf{A}\|_F$, is defined to be $\sqrt{\mathbf{trace}(\mathbf{A}^H\mathbf{A})}$. **62**

Exercise 111 Show that

$$\|\mathbf{A}\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|A_{i,j}|^2.$$

Exercise 112 Show that the Frobenius norm satisfies all the properties of a norm.

Induced matrix norm Let $\|\cdot\|_\alpha$ be a norm on $\mathbb{F}^n$ and let $\|\cdot\|_\beta$ be a norm on $\mathbb{F}^m$. On $\mathbb{F}^{m\times n}$ define the norm **63**

$$\|\mathbf{A}\|_{\alpha,\beta} = \sup_{\mathbf{0}\neq\mathbf{x}\in\mathbb{F}^n}\frac{\|\mathbf{Ax}\|_\beta}{\|\mathbf{x}\|_\alpha}.$$

Exercise 113 Show that $\|\cdot\|_{\alpha,\beta}$ satisfies all the properties of a norm.

Exercise 114 Show that

$$\|\mathbf{Ax}\|_\beta \leq \|\mathbf{A}\|_{\alpha,\beta}\|\mathbf{x}\|_\alpha.$$

Induced matrix p-norms For $\mathbf{A}\in\mathbb{F}^{m\times n}$ we define the p-norm of $\mathbf{A}$ to be **64**

$$\|\mathbf{A}\|_p = \sup_{\mathbf{0}\neq\mathbf{x}\in\mathbb{F}^n}\frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p}, \qquad 1\leq p\leq\infty.$$

Exercise 115 For $\mathbf{x}\in\mathbb{F}^{m\times 1}$ show that the vector p-norm and matrix p-norm give identical values.

Exercise 116 Show that for $\mathbf{A}\in\mathbb{F}^{m\times n}$

$$\|\mathbf{A}\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^{m}|A_{i,j}|.$$

Exercise 117 Show that for $\mathbf{A}\in\mathbb{F}^{m\times n}$

$$\|\mathbf{A}\|_\infty = \max_{1\leq i\leq m}\sum_{j=1}^{n}|A_{i,j}|.$$

Exercise 118 **Sub-multiplicative property**: show that

$$\|\mathbf{AB}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p.$$

Exercise 119 Establish the following inequalities for $\mathbf{A} \in \mathbb{F}^{m \times n}$

$$\begin{aligned}
\|\mathbf{A}\|_1 &\leq m\|\mathbf{A}\|_\infty \\
\|\mathbf{A}\|_\infty &\leq n\|\mathbf{A}\|_1 \\
\|\mathbf{A}\|_1 &\leq \sqrt{m}\|\mathbf{A}\|_2 \\
\|\mathbf{A}\|_2 &\leq \sqrt{n}\|\mathbf{A}\|_1.
\end{aligned}$$

Hint: The corresponding inequalities for vector norms might prove useful.

Exercise 120 Show that for $\mathbf{A} \in \mathbb{F}^{m \times n}$

$$\|\mathbf{A}\|_2 = \sup_{\substack{\mathbf{0} \neq \mathbf{y} \in \mathbb{F}^m \\ \mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n}} \frac{\left|\mathbf{y}^H \mathbf{A} \mathbf{x}\right|}{\|\mathbf{y}\|_2 \|\mathbf{x}\|_2}.$$

Exercise 121 Show that $\|\mathbf{A}\|_2 = \|\mathbf{A}^H\|_2$.

Exercise 122 Show that $\|\mathbf{AB}\|_F \leq \min\{\|\mathbf{A}\|_2\|\mathbf{B}\|_F, \|\mathbf{A}\|_F\|\mathbf{B}\|_2\}$.

Exercise 123 Show that $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$.

Exercise 124 Show that the Frobenius norm is sub-multiplicative.

Exercise 125 Show that for $\mathbf{A} \in \mathbb{F}^{m \times n}$

$$\|\mathbf{A}\|_p = \sup_{\substack{\mathbf{0} \neq \mathbf{y} \in \mathbb{F}^m \\ \mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n}} \frac{\left|\mathbf{y}^H \mathbf{A} \mathbf{x}\right|}{\|\mathbf{y}\|_q \|\mathbf{x}\|_p}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Exercise 126 Show that $\|\mathbf{A}\|_p = \|\mathbf{A}^H\|_q$ when $pq = p + q$.

An important, but little known result, is one of Holmgren's,

$$\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_\infty.$$

Exercise 127 Show that for $c > 0$,

$$xy \leq c\frac{x^2}{2} + \frac{1}{c}\frac{y^2}{2},$$

and that the lower bound is achieved for some $c \geq 0$ when $x, y \geq 0$.

Since, for $\mathbf{x} \in \mathbb{F}^n$ and $\mathbf{y} \in \mathbb{F}^m$,

$$\left| y^H \mathbf{A}\mathbf{x} \right| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{i,j}| \, |y_i| \, |x_j| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{i,j}| \left(c \frac{|y_i|^2}{2} + \frac{1}{c} \frac{|x_j|^2}{2} \right),$$

whence

$$\left| y^H \mathbf{A}\mathbf{x} \right| \leq \frac{c}{2} \|\mathbf{A}\|_\infty \|\mathbf{y}\|_2^2 + \frac{1}{2c} \|\mathbf{A}\|_1 \|\mathbf{x}\|_2^2.$$

Therefore, using the achievability of the lower-bound of exercise 127, we can conclude that

$$\frac{\left| y^H \mathbf{A}\mathbf{x} \right|}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} \leq \sqrt{\|\mathbf{A}\|_1 \|\mathbf{A}\|_\infty},$$

from which Holmgren's result follows.

Exercise 128 Why?

3.6 Riesz–Thorin

Holmgren's result is a special case of a result of M. Riesz. Due to an elegant proof of Thorin it is called the Riesz–Thorin interpolation theorem. We present a specialized version of the result.

Riesz-Thorin interpolation theorem **65**

$$\|\mathbf{A}\|_{p(a)} \le \|\mathbf{A}\|_{p_0}^{1-a}\|\mathbf{A}\|_{p_1}^{a}, \qquad \frac{1}{p(a)} = \frac{1-a}{p_0} + \frac{a}{p_1}, \qquad 0 \le a \le 1.$$

We give a brief and dirty review of the needed complex analysis. For the net few exercises *engineering proofs* are good enough, as a lot more work is needed to enable rigorous proofs.

Taylor series A formal series of the form **66**

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

is called a Taylor series about the point $a \in \mathbb{C}$.

Radius of Convergence The radious of convergence of a Taylor series $\sum_{n=0}^{\infty} a_n(z-a)^n$, is a number R, possibly infinite, such that **67**

$$\sum_{n=0}^{\infty} |a_n|\,|z-a|^n < \infty$$

whenever $|z-a| < R$.

Let Ω denote an open set in $\mathbb{C}$. We assume that the boundary of Ω is a piece-wise smooth curve that is simply connected.

Analytic A function f is said to be analytic in Ω, if at every point $a \in \Omega$ it has a Taylor series representation, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, with a non-zero radius of convergence. **68**

e^z Let **69**

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Exercise 129 Show that e^z is analytic in $\mathbb{C}$.

Exercise 130 Let Γ denote the circle $|z-a| = R$, such that $\Gamma \subset \Omega$. Let f be analytic in Ω. Show that

$$\int_\Gamma f(z)dz = 0.$$

Hint: Take $z - a = Re^{i\theta}$ and $dz = Rie^{i\theta}d\theta$ and write it as an ordinary integral over $0 \le \theta \le 2\pi$.

Exercise 131 Show that

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz.$$

This is called Cauchy's integral formula. *Hint*: Use a Taylor series expansion for f integrate term-by-term.

Exercise 132 Show that

$$|f(a)| \le \max_{|z-a|=R} |f(z)|.$$

Exercise 133 Show that $f(z)$ must attain its maximum (and minimum) at the boundary of Ω. This is called the **maximum principle**.

This is the end of the review, as all we needed was the maximum principle. You should be able to give complete proofs from now on.

For the rest of this section let Ω be the strip $0 \le \mathrm{Re}z \le 1$.

Exercise 134 Show that $|e^{\lambda z}|$, with real λ, must achieve its maximum and minimum in Ω (independently) on one of the lines $\mathrm{Re}(z) = 0$ or $\mathrm{Re}(z) = 1$. This does not require the maximum principle.

Exercise 135 Show that $|\sum_{k=1}^N z_k e^{\lambda_k z}|$ with real λ_k achieves its maximum on one of the lines $\mathrm{Re}(z) = 0$ or $\mathrm{Re}(z) = 1$.

Hadamard's three lines lemma Let $f(z)$ be analytic in an open set containing Ω. Let **70**

$$F(a) = \sup_y |f(a+iy)|, \qquad 0 \le a \le 1.$$

Then

$$F(a) \le F^{1-a}(0)F^a(1).$$

Proof of three lines lemma. Let

$$\phi(z) = f(z)e^{z\log\frac{F(0)}{F(1)}}.$$

Clearly ϕ is analytic in an open set containing Ω. By the maximum principle $|\phi(z)| \leq F(0)$ on Ω. Therefore

$$|f(a+iy)|\, e^{a \log \frac{F(0)}{F(1)}} \leq F(0),$$

and from this the three lines lemma follows.

Exercise 136 Why? □

We note that

$$\|\mathbf{A}\|_p = \sup_{\mathbf{x},\mathbf{y} \neq \mathbf{0}} \frac{\left|\mathbf{y}^H \mathbf{A}\mathbf{x}\right|}{\|\mathbf{y}\|_q \|\mathbf{x}\|_p}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Let

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1},$$

and

$$\frac{1}{p(z)} + \frac{1}{q(z)} = 1.$$

Observe that

$$\frac{1}{q(z)} = \frac{1-z}{q(0)} + \frac{z}{q(1)}.$$

Exercise 137 Prove it.

Let $\|\mathbf{x}\|_{p(a)} = \|\mathbf{y}\|_{q(a)} = 1$. Let $x_k = |x_k| e^{i\psi_k}$ and $y_k = |y_k| e^{i\theta_k}$. Define

$$x_k(z) = |x_k|^{\frac{p(a)}{p(z)}} e^{i\psi_k} \qquad \text{and} \qquad y_k(z) = |y_k|^{\frac{q(a)}{q(z)}} e^{i\theta_k}.$$

Define

$$f(z) = \mathbf{y}^H(z)\, \mathbf{A}\, \mathbf{x}(z).$$

Note that $1/p(z)$ and $1/q(z)$ are linear functions in z, and hence analytic in z. Therefore $\mathbf{x}(z)$ and $\mathbf{y}(z)$, and hence $f(z)$, are also analytic functions of z.

Exercise 138 Prove it.

As before let $F(a) = \sup_y |f(a+iy)|$. Then it is true that

$$F(0) \leq \|\mathbf{A}\|_{p_0} \qquad \text{and} \qquad F(1) \leq \|\mathbf{A}\|_{p_1}. \tag{3.2}$$

To prove these we first observe that

$$\mathrm{Re}\left(\frac{1}{p(x+iy)}\right) = \frac{1-x}{p_0} + \frac{x}{p_1} = \frac{1}{p(x)}.$$

Exercise 139 Prove it.

Hence it also follows that

$$\mathrm{Re}\left(\frac{1}{q(x+iy)}\right) = \frac{1}{q(x)}.$$

Therefore we can conclude that $\|\mathbf{x}(\alpha+i\beta)\|_{p(\alpha)} = \|\mathbf{x}(\alpha)\|_{p(\alpha)}$. Similarly $\|\mathbf{y}(\alpha + i\beta)\|_{q(\alpha)} = \|\mathbf{y}(\alpha)\|_{q(\alpha)}$.

Exercise 140 Prove it.

Next we note that $\|\mathbf{x}(0)\|_{p_0}^{p_0} = \|\mathbf{x}(a)\|_{p(a)}^{p(a)} = 1 = \|\mathbf{x}(1)\|_{p_1}^{p_1}$. Similarly $\|\mathbf{y}(0)\|_{q(0)}^{q(0)} = \|\mathbf{y}(a)\|_{q(a)}^{q(a)} = 1 = \|\mathbf{y}(1)\|_{q(1)}^{q(1)}$.

Exercise 141 Prove it.

From this it follows, using Hölder's inequality, that

$$F(0) = \sup_\beta |f(i\beta)| \leq \sup_\beta \|\mathbf{y}(i\beta)\|_{q(0)} \|\mathbf{A}\|_{p_0} \|\mathbf{x}(i\beta)\|_{p_0} = \|\mathbf{A}\|_{p_0}.$$

Similarly we can establish that

$$F(1) \leq \|\mathbf{A}\|_{p_1}.$$

Now choose $\mathbf{x}$ and $\mathbf{y}$ such that $f(a) = \|\mathbf{A}\|_{p(a)}$, in addition to the fact that that $\|\mathbf{x}\|_{p(a)} = \|\mathbf{y}\|_{q(a)} = 1$. Then it follows that

$$F(a) = \sup_b |f(a+ib)| \leq \|\mathbf{A}\|_{p(a)} = |f(a)| \leq F(a).$$

Now apply the three lines lemma to obtain the Riesz–Thorin theorem.

Exercise 142 Do so.

For finite–dimensional matrices Holmgren's result is more than sufficient in practice. The Riesz–Thorin result exhibits its power in the infinite–dimensional case, where one or both of the 1–norm and the ∞–norm may be infinite.

3.7 Perturbed inverses

We will now show that $\mathbf{A}^{-1}$ is a continuous functions of its entries. There are several ways to establish this fact. We will take a route via Neumann's theorem that is useful in its own right.

Convergence of matrix sequences Let $\mathbf{A}_n$ for $n = 1, 2, \ldots$, denote a sequence of $m \times n$ matrices. We say that $\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$, if every component of $\mathbf{A}_n$ converges to the corresponding component of $\mathbf{A}$. In other words convergence of a matrix sequence is defined component-wise. **71**

Exercise 143 Show that $\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$ iff $\lim_{n\to\infty} \|\mathbf{A}_n - \mathbf{A}\| = 0$, for any valid matrix norm. Note that this not true for matrices of infinite size.

Convergence of matrix sums We say that $\sum_{n=1}^{\infty} \mathbf{A}_n = \mathbf{A}$ if $\lim_{n\to\infty} \mathbf{S}_n = \mathbf{A}$, with $\mathbf{S}_N = \sum_{n=1}^{N} \mathbf{A}_n$. **72**

Just like infinite sums of numbers, convergence of infinite matrix sums can be delicate.

Exercise 144 **Riemann's theorem**. Show that by re-ordering the sum $\sum_{n=1}^{\infty}(-1)^n/n$ you can make it converge to any real number.

This cannot happen if the series converges absolutely. Geometrically if you think of the series as a string with marks on it corresponding to the individual terms, bad things can happen only if the string has infinite length.

Absolute convergence We say that $\sum_{n=1}^{\infty} \mathbf{A}_n$ converges absolutely if $\sum_{n=1}^{\infty} \|\mathbf{A}\|_n < \infty$, for some matrix norm. **73**

Exercise 145 Show that if $\sum_{n=1}^{\infty} \|\mathbf{A}\|_n < \infty$ then there exists a finite matrix $\mathbf{A}$ such $\sum_{n=1}^{\infty} \mathbf{A}_n = \mathbf{A}$.

Neumann's Theorem Let $\mathbf{A}$ be a square matrix such that $\|\mathbf{A}\| < 1$ for some induced matrix norm. It then follows that **74**

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{n=0}^{\infty} \mathbf{A}^n,$$

with absolute convergence of the series on the right.

Proof. This is just the matrix version of the geometric series.

Exercise 146 Show that for $|z| < 1$, $(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n$, with the series converging absolutely.

Exercise 147 Show that $\sum_{n=1}^{\infty} \mathbf{A}^n$ converges absolutely since $\|\mathbf{A}_n\| < 1$.

The only question is whether it converges to $(\mathbf{I}-\mathbf{A})^{-1}$? First we prove the required inverse exists. Suppose it does not. Then there exists a vector $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ such that $\mathbf{Ax} = \mathbf{x}$. (Why?)

Exercise 148 Show that this implies that $\|\mathbf{A}\| \geq 1$, which is a contradiction.

It follows that $\mathbf{I} - \mathbf{A}$ is invertible.

Exercise 149 Suppose $\sum_{n=1}^{\infty} \mathbf{A}_n$ and $\sum_{n=1}^{\infty} \mathbf{B}_n$ are two absolutely converging matrix series. Show that

- $\sum_{n=1}^{\infty} \mathbf{A}_n + \sum_{n=1}^{\infty} \mathbf{B}_n = \sum_{n=1}^{\infty} (\mathbf{A}_n + \mathbf{B}_n)$
- $\mathbf{C} \sum_{n=1}^{\infty} \mathbf{A}_n = \sum_{n=1}^{\infty} \mathbf{C}\mathbf{A}_n$

Exercise 150 Show that $(\mathbf{I} - \mathbf{A}) \sum_{n=0}^{\infty} \mathbf{A}^n = \mathbf{I}$. □

Exercise 151 Show that if $\mathbf{A} = \sum_{n=1}^{\infty} \mathbf{A}_n$ then $\|\mathbf{A}\| \leq \sum_{n=1}^{\infty} \|\mathbf{A}_n\|$.

Exercise 152 Show that if $\|\mathbf{A}\| < 1$ for some induced matrix norm then $\|(\mathbf{I}-\mathbf{A})^{-1}\| \leq (1-\|\mathbf{A}\|)^{-1}$.

Exercise 153 Let $\|\mathbf{A}^{-1}\|\|\mathbf{E}\| < 1$ for some induced matrix norm. Show that $\mathbf{A} + \mathbf{E}$ is non-singular and that

$$\frac{\|(\mathbf{A}+\mathbf{E})^{-1} - \mathbf{A}^{-1}\|}{\|\mathbf{A}^{-1}\|} \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \frac{\|\mathbf{E}\|}{\|\mathbf{A}\|} \frac{1}{1 - \|\mathbf{A}^{-1}\|\|\mathbf{E}\|}.$$

The factor $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ is called the **condition number** of the matrix $\mathbf{A}$ and it is the amplification factor for the norm-wise relative error in $\mathbf{A}^{-1}$ due to relative norm-wise perturbations in $\mathbf{A}$. In general, linear systems with large condition numbers are difficult to solve accurately on floating-point machines. It is something that one should always be aware of.

4 Orthogonality

The fact that the vector 2-norms are related to matrix multiplication leads to a powerful algebraic technique.

4.1 Unitary Matrices

Orthonormal A set of column vectors $\mathbf{v}_i$ is said to be orthonormal if $\|\mathbf{v}_i\|_2 = 1$ and $\mathbf{v}_i^H \mathbf{v}_j = 0$ for $i \neq j$. **75**

Unitary Matrix A square matrix $\mathbf{U}$ is said to be unitary if $\mathbf{U}^H \mathbf{U} = \mathbf{I}$. **76**

Orthogonal Matrix A real unitary matrix is called an orthogonal matrix. **77**

Exercise 154 Show that if the matrix $\mathbf{U}$ is unitary then $\mathbf{U}\mathbf{U}^H = I$.

Exercise 155 Show that the rows of a unitary matrix form an orthonormal set.

Exercise 156 Show that the columns of a unitary matrix form an orthonormal set.

Exercise 157 Show that the product of two unitary matrices is unitary.

Exercise 158 Let $\mathbf{U}$ be a $n \times n$ unitary matrix. Show that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{y}^H \mathbf{x} = (\mathbf{U}\mathbf{y})^H (\mathbf{U}\mathbf{x})$. Therefore unitary transforms preserve inner products. Conclude that unitary transforms preserve 2-norms and angles of column vectors.

Exercise 159 Show that $\|\mathbf{UAV}\|_F = \|\mathbf{A}\|_F$, if $\mathbf{U}$ and $\mathbf{V}$ are unitary transforms.

Exercise 160 Show that $\|\mathbf{UAV}\|_2 = \|\mathbf{A}\|_2$, if $\mathbf{U}$ and $\mathbf{V}$ are unitary transforms.

Exercise 161 Show that permutation matrices are orthogonal matrices.

Householder Transform A matrix of the form $\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^H}{\mathbf{v}^H\mathbf{v}}$ is called a Householder transform, where $\mathbf{v}$ is a non-zero column vector. **78**

Exercise 162 Show that a Householder transform is a Hermitian unitary matrix.

Exercise 163 Consider the Householder transform $\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^H}{\mathbf{v}^H\mathbf{v}}$. Show that $\mathbf{H}\mathbf{v} = -\mathbf{v}$. Show that if $\mathbf{x}^H \mathbf{v} = 0$, then $\mathbf{H}\mathbf{x} = \mathbf{x}$.

Exercise 164 Explain why the Householder transform is called an elementary reflector.

Exercise 165 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show, by construction, that there is a Householder transform $\mathbf{H}$ such that $\mathbf{H}\mathbf{x} = \mathbf{y}$, if $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$.

Elementary Gauss and Householder transforms are the main ingredients for the algorithmic construction of matrix decompositions.

4.2 The Singular Value Decomposition

Or, the SVD, is the sledge-hammer that solves all problems in matrix analysis (or something like that).

Exercise 166 Show that for $\mathbf{A} \in \mathbb{C}^{m \times n}$

$$\|\mathbf{A}\|_2 = \sup_{\|\mathbf{x}\|_2 = \| y \|_2 = 1} |\mathbf{y}^H \mathbf{A}\mathbf{x}|.$$

Exercise 167 Since the unit spheres for the 2-norm in $\mathbb{C}^n$ and $\mathbb{C}^m$ are compact, and matrix products are continuous functions, show that there exists $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$ such that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, and $\mathbf{A}\mathbf{x} = \|\mathbf{A}\|_2 \mathbf{y}$.

SVD For every $m \times n$ matrix $\mathbf{A}$ there exist unitary matrices $\mathbf{U}$ and $\mathbf{V}$ and a matrix **79**
$\Sigma \in \mathbb{R}^{m \times n}$ of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots \\ 0 & \sigma_2 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix},$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$, such that $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$.

Proof. Let $\|\mathbf{x}\|_2 = 1 = \|\mathbf{y}\|_2$ such that $\mathbf{A}\mathbf{x} = \|\mathbf{A}\|_2\mathbf{y}$. Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two Householder transforms such that $\mathbf{H}_1\mathbf{x} = \mathbf{e}_1$ and $\mathbf{H}_2\mathbf{y} = \mathbf{e}_1$, where $\mathbf{e}_i$ denotes column i of the appropriate identity matrix. Now we claim that

$$\mathbf{H}_2\mathbf{A}\mathbf{H}_1^H = \begin{pmatrix} \|\mathbf{A}\|_2 & \mathbf{b}^H \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Exercise 168 Prove it.

Next we note that $\mathbf{b} = 0$. To prove this first note that $\|\mathbf{H}_2\mathbf{A}\mathbf{H}_1^H\|_2 = \|\mathbf{A}\|_2$ since $\mathbf{H}_1$ and $\mathbf{H}_2$ are unitary.

Exercise 169 Show that

$$\frac{\left\| \begin{pmatrix} \|\mathbf{A}\|_2 & \mathbf{b}^H \\ \mathbf{0} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \|\mathbf{A}\|_2 \\ \mathbf{b} \end{pmatrix} \right\|_2}{\left\| \begin{pmatrix} \|\mathbf{A}\|_2 \\ \mathbf{b} \end{pmatrix} \right\|_2} \geq \sqrt{\|\mathbf{A}\|_2^2 + \|\mathbf{b}\|_2^2}.$$

But this would imply that $\|\mathbf{H}_2\mathbf{A}\mathbf{H}_1^H\|_2 > \|\mathbf{A}\|_2$ unless $\mathbf{b} = \mathbf{0}$. Hence we have that

$$\mathbf{H}_2\mathbf{A}\mathbf{H}_1^H = \begin{pmatrix} \|\mathbf{A}\|_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Clearly we can take $\|\mathbf{A}\|_2 = \sigma_1$ in the proof. To finish we can proceed by induction. Assuming that we have SVD's for all matrices of size $(m-1)\times(n-1)$ and smaller, let $\mathbf{C} = \mathbf{U}_1\Sigma_1\mathbf{V}_1^H$ be the SVD of $\mathbf{C}$. Then it is clear that

$$\mathbf{A} = \underbrace{\mathbf{H_2}^H \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{pmatrix}}_{\mathbf{U}} \underbrace{\begin{pmatrix} \|\mathbf{A}\|_2 & \mathbf{0} \\ \mathbf{0} & \Sigma_1 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1 \end{pmatrix}^H \mathbf{H}_1}_{\mathbf{V}^H}.$$

Exercise 170 Check that $\mathbf{U}$ and $\mathbf{V}$ in the above formula are unitary and that Σ has the desired diagonal structure with real non-negative entries on the main diagonal.

For the base case of the induction it is sufficient to write down the SVD of an empty (either rows or columns) matrix

$$\mathbf{A} = \mathbf{I}\,\mathbf{0}\,\mathbf{I}^H.$$

Exercise 171 Check that this base case is sufficent.

The only thing left to check is that the diagonal entries in Σ are in decreasing order. The easy way out is to say that if they are not in decreasing order then we can apply two permutation matrices from the left and right to correct the order and note that permutations are unitary. But it is more informative to note instead that $\|\mathbf{C}\|_2 \leq \|\mathbf{A}\|_2$.

This follows from the following more general fact.

Exercise 172 Show that

$$\left\| \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \right\|_p = \max\left(\|\mathbf{A}\|_p, \|\mathbf{B}\|_p\right),$$

for $1 \leq p \leq \infty$. □

The columns of $\mathbf{U}$ are called the **left singular vectors** of $\mathbf{A}$, while the columns of $\mathbf{V}$ are called the **right singular vectors**. The σ_i are called the **singular values** of $\mathbf{A}$.

Exercise 173 Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & \cdots \\ 0 & a_{22} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}_{m\times n}.$$

Show that $\|\mathbf{A}\|_p = \max\limits_{1\le i\le \min(m,n)} |a_{ii}|$ for $1 \le p \le \infty$.

Exercise 174 Show that $\|\mathbf{A}\|_2 = \sigma_1$ and $\|\mathbf{A}\|_F^2 = \sigma_1^2 + \cdots + \sigma_{\min(m,n)}^2$, where σ_i are the singular values of $\mathbf{A}$.

4.3 Orthogonal Subspaces

Orthogonal subspaces Two subspaces $\mathcal{U}$ and $\mathcal{W}$ of $\mathbb{F}^n$ are said to be orthogonal to each other if every vector 80
in $\mathcal{U}$ is orthogonal to every vector in $\mathcal{W}$. This is denoted by $\mathcal{U} \perp \mathcal{W}$.

Exercise 175 Show that $\mathcal{U} \cap \mathcal{W} = \{0\}$ if $\mathcal{U} \perp \mathcal{W}$.

Orthogonal Complement The orthogonal complement of the set $\mathcal{U}$ is the set of all vectors that are orthogonal 81
to all vectors in $\mathcal{U}$. It is denoted as $\mathcal{U}^\perp$.

Exercise 176 Show that $\mathcal{U} \perp \mathcal{U}^\perp$.

Exercise 177 Let $\mathbf{U} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}$ be an $n \times n$ unitary matrix. Show that

- The columns of $\mathbf{U}_1$ form an orthonormal basis for $\mathcal{R}(\mathbf{U}_1)$
- $\mathcal{R}(\mathbf{U}_1) = \mathcal{R}(\mathbf{U}_2)^\perp$
- $(\mathcal{R}(\mathbf{U}_1)^\perp)^\perp = \mathcal{R}(\mathbf{U}_1)$
- $\mathcal{R}(\mathbf{U}_1) \oplus \mathcal{R}(\mathbf{U}_1)^\perp = \mathbb{C}^n$

Let the SVD of $\mathbf{A}$ be partitioned as follows

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix}^H,$$

where $\Sigma_1 \in \mathbb{R}^{r\times r}$ is a non-singular diagonal matrix. That is, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Exercise 178 Show that $\mathbf{A} = \mathbf{U}_1\Sigma_1\mathbf{V}_1^H$.

This is sometimes called the **economy SVD** of $\mathbf{A}$.

Exercise 179 Show that $\mathbf{U}_1$ and $\mathbf{V}_1$ are full column-rank matrices (rank r).

The SVD gives a full description of the geometry of the four fundamental subspaces associated with the matrix $\mathbf{A}$.

Exercise 180 Show that

- $\mathbf{A}\mathbf{V}_1 = \mathbf{U}_1\Sigma_1$
- $\mathbf{A}\mathbf{V}_2 = \mathbf{0}$
- $\mathbf{U}_1^H\mathbf{A} = \Sigma_1\mathbf{V}_1^H$

- $\mathbf{U}_2^H \mathbf{A} = \mathbf{0}$
- $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{U}_1)$
- $\mathcal{R}(\mathbf{A}^H) = \mathcal{R}(\mathbf{V}_1)$
- $\mathcal{R}(\mathbf{V}_2) = \mathcal{N}(\mathbf{A})$
- $\mathcal{R}(\mathbf{U}_2) = \mathcal{N}(\mathbf{A}^H)$
- $\mathcal{R}(\mathbf{A}^H) = \mathcal{N}(\mathbf{A})^\perp$
- $\mathcal{R}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^H)$
- $\mathbf{rank}(\mathbf{A}) = r$, the number of non-zero singular values of $\mathbf{A}$

Exercise 181 Let $\mathcal{U}$ denote a subspace of $\mathbb{C}^n$. Construct an orthonormal basis for $\mathcal{U}$ from one of its basis using the SVD.

Exercise 182 Let $\mathcal{U}$ be a subspace of $\mathbb{C}^n$. Show that

- $\mathcal{U}^{\perp\perp} = \mathcal{U}$
- $\mathcal{U} \oplus \mathcal{U}^\perp = \mathbb{C}^n$

Orthogonal Projector The **orthogonal projector** onto the subspace $\mathcal{U}$ is defined to be a linear operator $P_{\mathcal{U}}$ with the following properties **82**

- $\mathcal{N}(P_{\mathcal{U}}) = \mathcal{U}^\perp$
- $P_{\mathcal{U}}\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$

Exercise 183 Show that orthogonal projectors are idempotent: $P_{\mathcal{U}}^2 = P_{\mathcal{U}}$.

Exercise 184 Show that $P_{\mathcal{U}}$ is unique for a given $\mathcal{U}$.

Exercise 185 Let $\mathbf{U} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}$ be a unitary matrix. Show that $\mathbf{U}_1\mathbf{U}_1^H = P_{\mathcal{R}(\mathbf{U}_1)}$.

Exercise 186 Show that orthogonal projectors are Hermitian.

Exercise 187 Construct an idempotent matrix that is not an orthogonal projector. These are called **oblique projectors**.

Exercise 188 Let $\mathbf{P}$ be a Hermitian idempotent matrix. Show that $\mathbf{P} = P_{\mathcal{R}(\mathbf{P})}$.

Exercise 189 Let $\mathcal{U}$ be a subspace of $\mathbb{C}^n$. Show that every $\mathbf{x} \in \mathbb{C}^n$ has a unique decomposition of the form $\mathbf{x} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in \mathcal{U}$ and $\mathbf{w} \in \mathcal{U}^\perp$. *Hint*: $\mathbf{u} = P_{\mathcal{U}}\mathbf{x}$.

Exercise 190 Show that

$$\min_{\mathbf{u}\in\mathcal{U}} \|\mathbf{x} - \mathbf{u}\|_2 = \|\mathbf{x} - P_{\mathcal{U}}\mathbf{x}\|_2.$$

4.4 Minimum norm least-squares solution

The **LU** factorization solved completely the question of finding all solutions of the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is unknown. However there is something unsatisfactory in that solution. Generically, skinny systems will almost surely have no solutions, while fat systems will almost surely have infinitely many solutions. Since both these cases are frequent in engineering a more informative approach is necessary.

Exercise 191 Let $\mathbf{x}_{\text{LS}}$ be such that

$$\min_{\mathbf{y}} \|\mathbf{A}\mathbf{y} - \mathbf{b}\|_2 = \|\mathbf{A}\mathbf{x}_{\text{LS}} - \mathbf{b}\|_2.$$

Show that $\mathbf{A}\mathbf{x}_{\text{LS}} = P_{\mathcal{R}(\mathbf{A})}\mathbf{b}$, and hence unique. Give an example where $\mathbf{x}_{\text{LS}}$ is not unique.

Let

$$X_{\text{LS}} = \{\mathbf{x} : \min_{\mathbf{y}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{y} - \mathbf{b}\|_2 = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2\}.$$

Affine Linear A subset X of a vector space $\mathcal{V}$ is said to be affine linear if there exists a vector $v \in \mathcal{V}$ such that the set $\{x - v : x \in X\}$ is a subspace. **83**

Exercise 192 Show that X_{LS} is a affine linear set.

Exercise 193 Show that there is a unique solution to

$$\min_{\mathbf{u}\in X} \|\mathbf{x} - \mathbf{u}\|_2,$$

where X is an affine linear set. *Hint*: Exercise 190.

Minimum Norm Least Squares solution Let **84**

$$\mathbf{x}_{\text{MNLS}} = \text{argmin}_{\mathbf{x}\in X_{\text{LS}}} \|\mathbf{x}\|_2.$$

Then $\mathbf{x}_{\text{MNLS}}$ is called the minimum norm least squares solution of the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Let $\mathbf{A} = \mathbf{U}_r\Sigma_r\mathbf{V}_r^H$ denote the economy SVD of $\mathbf{A}$. Then

$$\mathbf{x}_{\text{MNLS}} = \mathbf{V}_r\Sigma_r^{-1}\mathbf{U}_r^H\mathbf{b}.$$

Exercise 194 Prove it.

Pseudo-inverse Let **85**

$$\Sigma = \begin{pmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

with Σ_r a non-singular diagonal matrix. Then we define the pseudo-inverse of Σ (denoted by superscript $\dagger$) as

$$\Sigma^\dagger = \begin{pmatrix} \Sigma_r^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

More generally, if $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$ is the SVD of $\mathbf{A}$ we then define $\mathbf{A}^\dagger = \mathbf{V}\Sigma^\dagger\mathbf{U}^H$.

The above definition may be ambiguous since the SVD of $\mathbf{A}$ is not unique.

Exercise 195 Show that $\mathbf{A}^\dagger = \mathbf{V}_r\Sigma_r^{-1}\mathbf{U}_r^H$, using the economy SVD of $\mathbf{A}$.

Therefore $\mathbf{x}_{\text{MNLS}} = \mathbf{A}^\dagger\mathbf{b}$. This can be used to define the pseudoinverse uniquely.

Exercise 196 Show that

- $\mathbf{A}\mathbf{A}^\dagger = P_{\mathcal{R}(\mathbf{A})}$
- $\mathbf{A}^\dagger\mathbf{A} = P_{\mathcal{R}(\mathbf{A}^H)}$
- $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$
- $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$

Roger Penrose showed that the pseudo-inverse is the unique solution to these four equations. However, we will take a different path.

Linear Map A map $A : \mathcal{V} \to \mathcal{W}$, between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ over the field $\mathbb{F}$ is said to be linear if $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in \mathcal{V}$. **86**

Matrix Representation Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between two vector spaces. Let $v_1, \cdots, v_n$ be a basis for $\mathcal{V}$. Let $w_1, \cdots, w_m$ be a basis for $\mathcal{W}$. Define the mn unique numbers $\mathbf{A}_{ij}$ by the equation $A(v_j) = \sum_{j=1}^{m} w_i \mathbf{A}_{ij}$. Then we call $\mathbf{A}$ the matrix representation of A for the given bases. **87**

Exercise 197 Why is $\mathbf{A}$ unique?

Exercise 198 Suppose $x \in \mathcal{V}$, and $b \in \mathcal{W}$, have the representations $x = \sum_{j=1}^{n} \mathbf{x}_j v_j$ and $b = \sum_{i=1}^{m} \mathbf{b}_i w_i$, and $A(x) = b$. Then show that $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Exercise 199 Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the field $\mathbb{F}$. Let $A : \mathcal{U} \to \mathcal{V}$ and $B : \mathcal{V} \to \mathcal{W}$ be two linear maps. Show that $B \circ A : \mathcal{U} \to \mathcal{W}$ is a linear map.

Exercise 200 If fixed bases are used for $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, then show that $\mathbf{BA}$ is a matrix representation for $B \circ A$.

Exercise 201 Show that $\mathbf{A} \in \mathbb{C}^{m \times n}$ is a one-to-one onto linear map from $\mathcal{R}(\mathbf{A}^H)$ to $\mathcal{R}(\mathbf{A})$. Call this map $B : \mathcal{R}(\mathbf{A}^H) \to \mathcal{R}(\mathbf{A})$.

Exercise 202 Show that in the appropriate bases for $\mathcal{R}(\mathbf{A}^H)$ and $\mathcal{R}(\mathbf{A})$, Σ_r is a matrix representation of B.

Exercise 203 Define the map $C : \mathbb{C}^n \to \mathbb{C}^m$ as follows: $C(\mathbf{b}) = B^{-1}(P_{\mathcal{R}(\mathbf{A})}\mathbf{b})$. Show that $\mathbf{A}^\dagger$ is a matrix representation of C.

This shows that the pseudo-inverse is uniquely defined.

Exercise 204 Why?

4.5 Problems

The SVD usually costs about 10 times as much as an **LU** fcatorization. A good substitute is the **QR** factorization.

Problem 5 Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ with $m \geq n$. Show that theres exists a unitary matrix $\mathbf{Q}$ such that

$$\mathbf{A} = \mathbf{Q}\begin{pmatrix}\mathbf{R}\\ \mathbf{0}\end{pmatrix},$$

where $\mathbf{R}$ is upper triangular with non-negative diagonal entries. *Hint*: This is similar to the construction of the SVD, but simpler.

Problem 6 Let $\mathbf{A}$ be a full column-rank matrix. Show that

$$\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \begin{pmatrix}\mathbf{R}^{-1} & \mathbf{0}\end{pmatrix}\mathbf{Q}^H$$

Problem 7 Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ with $n \geq m$. Show that theres exists a unitary matrix $\mathbf{Q}$ such that

$$\mathbf{A} = \begin{pmatrix}\mathbf{L} & \mathbf{0}\end{pmatrix}\mathbf{Q},$$

where $\mathbf{L}$ is lower triangular with non-negative diagonal entries

Problem 8 Let $\mathbf{A}$ be a full row-rank matrix. Show that

$$\mathbf{A}^\dagger = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1} = \mathbf{Q}^H\begin{pmatrix}\mathbf{L}^{-1}\\ \mathbf{0}\end{pmatrix}$$

Problem 9 Find the shortest distance between two infinite straight lines in $\mathbb{R}^n$.

Problem 10 Show that $\|\mathbf{A}\|_F \leq \sqrt{\mathbf{rank}(\mathbf{A})}\|\mathbf{A}\|_2$.

5 Spectral Theory

In principle we have covered everything for solving systems of linear equations. However, our techniques (meaning **LU** factorization) do not generalize (yet?) to infinite-number of equations. A host of different techniques have been developed for handling this case. Spectral methods are among the most powerful of these.

Examples of infinite number of equations include differential and difference equations, and it was in their analysis that spectral theory was first born.

5.1 Spectral Decompositions

In this section, unless mentioned otherwise, all matrices will be assumed to be square.

Exercise 205 Show that $\mathbf{dim}(\mathbb{C}^{n\times n}) = n^2$.

We will assume that $\mathbf{A}^0 = \mathbf{I}$ and that $\mathbf{A}^{k+1} = \mathbf{A}\mathbf{A}^k$ for $k \geq 1$. If $\mathbf{A}$ is invertible we will define $\mathbf{A}^{-k} = (\mathbf{A}^{-1})^k$ for $k \geq 0$.

Polynomial of a matrix Let $p(x) = \sum_{n=0}^{N} a_n x^n$. Define $p(\mathbf{A}) = \sum_{n=0}^{N} a_n \mathbf{A}^n$. **88**

For this definition to be useful, we need to ensure that different ways of defining the same polynomial yield the same value when evaluated at a matrix. For example, if q and r are polynomials, we would like that $q(\mathbf{A})r(\mathbf{A}) = r(\mathbf{A})q(\mathbf{A}) = (qr)(\mathbf{A})$ for all square matrices $\mathbf{A}$.

Exercise 206 Prove that it is so.

Lemma 1 For every square matrix $\mathbf{A}$ there is a complex number λ such that $\mathbf{A} - \lambda\mathbf{I}$ is singular.

Proof.

Exercise 207 For a given $\mathbf{A} \in \mathbb{C}^{n\times n}$, show that there exist $n^2 + 1$ complex numbers α_i, for $0 \leq i \leq n^2$, not all zero, such that

$$\sum_{i=0}^{n^2} \alpha_i \mathbf{A}^i = \mathbf{0}.$$

Let $p(x) = \sum_{i=0}^{n^2} \alpha_i x^i$ be the corresponding polynomial. Let $M \geq 1$ be its degree. (Why not 0?). It is well known that p can be factored as

$$p(x) = \prod_{i=0}^{M} (x - \mu_i),$$

for M complex numbers μ_i (possibly indistinct). It follows that

$$p(\mathbf{A}) = \prod_{i=0}^{M} (\mathbf{A} - \mu_i \mathbf{I}) = \mathbf{0}.$$

Exercise 208 Make sure you understand why exactly this is true.

Since the product of two square non-singular matrices is non-singular (why?) it follows that there exists some i for which $\mathbf{A} - \mu_i \mathbf{I}$ is singular. □

Schur decomposition For every square matrix $\mathbf{A}$ there exists a unitary matrix $\mathbf{Q}$ and an upper-triangular matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{R}\mathbf{Q}^H$. **89**

This is the computationally stable factorization in spectral theory, and hence of great practical significance.

Proof. The proof is by induction. For 1×1 matrices the theorem is obviously true: $\mathbf{A} = \mathbf{I}\mathbf{A}\mathbf{I}^H$. Assume it is true for all matrices of size $(n-1) \times (n-1)$ or smaller. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$. Let λ be a complex number such that $\mathbf{A} - \lambda \mathbf{I}$ is singular. Let $\mathbf{v} \in \mathcal{N}(\mathbf{A} - \lambda \mathbf{I})$ be of unit length: $\|\mathbf{v}\|_2 = 1$. Choose a Householder transform $\mathbf{H}$ such that $\mathbf{H}\mathbf{v} = \mathbf{e}_1$ (where $\mathbf{e}_i$ denotes column i of the identity matrix). Then it is easy to see that

$$\mathbf{H}\mathbf{A}\mathbf{H}^H = \begin{pmatrix} \lambda & \mathbf{b}^H \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Exercise 209 Prove it.

By the inductive assumption $\mathbf{C} = \mathbf{Q}_1 \mathbf{R}_1 \mathbf{Q}_1^H$, where $\mathbf{Q}_1$ is unitary and $\mathbf{R}_1$ is upper triangular. It follows that

$$\mathbf{A} = \underbrace{\mathbf{H}^H \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_1 \end{pmatrix}}_{\mathbf{Q}} \underbrace{\begin{pmatrix} \lambda & \mathbf{b}^H \mathbf{Q}_1 \\ \mathbf{0} & \mathbf{R}_1 \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_1 \end{pmatrix}^H \mathbf{H}^H,}_{\mathbf{Q}^H}$$

Exercise 210 Prove it. □

In general the diagonal entries of $\mathbf{R}$ are arbitrary complex numbers. However, we can impose some order on them that is of significance.

Exercise 211 Suppose $\mathbf{A} = \mathbf{V}\mathbf{B}\mathbf{V}^{-1}$. Show that $\textbf{trace}(\mathbf{A}) = \textbf{trace}(\mathbf{B})$.

Lemma 2 Let

$$\mathbf{R} = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}.$$

There exists a unitary matrix $\mathbf{Q}$ such that

$$\mathbf{Q}\mathbf{R}\mathbf{Q}^H = \begin{pmatrix} \lambda_2 & \beta \\ 0 & \lambda_1 \end{pmatrix}.$$

Proof. There is nothing to prove if $\lambda_1 = \lambda_2$. So we consider the case $\lambda_1 \neq \lambda_2$. Choose $\mathbf{v}$ such that $\mathbf{R}\mathbf{v} = \lambda_2 \mathbf{v}$ and $\|\mathbf{v}\|_2 = 1$.

Exercise 212 Find $\mathbf{v}$ explicitly.

Choose a Householder transform $\mathbf{H}$ such that $\mathbf{H}\mathbf{v} = \mathbf{e}_1$.

Exercise 213 Find $\mathbf{H}$ explicitly.

Then we can choose $\mathbf{Q} = \mathbf{H}$.

Exercise 214 Prove it. □

Strictly upper triangular A strictly upper triangular matrix is an upper triangular matrix with zero entries on the diagonal. **90**

Lemma 3 For every square matrix $\mathbf{A}$ there is an unitary matrix $\mathbf{Q}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{R}\mathbf{Q}^H$ with

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \cdots & \mathbf{R}_{1M} \\ \mathbf{0} & \mathbf{R}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{R}_{M-1,M} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R}_{MM} \end{pmatrix},$$

where $\mathbf{R}_{ii} = \lambda_i \mathbf{I} + \widetilde{\mathbf{R}}_{ii}$ with $\widetilde{\mathbf{R}}_{ii}$ being a strictly upper triangular matrix, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Proof. The proof follows from a simple observation. Suppose two adjacent diagonal entries in the matrix $\mathbf{R}$ from the Schur decomposition are distinct

$$
\mathbf{R} = \begin{pmatrix}
* & * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & \ddots & \ddots & & & & & \vdots \\
\vdots & \ddots & * & * & & & & \vdots \\
\vdots & & \ddots & \lambda_1 & \alpha & & & \vdots \\
\vdots & & & \ddots & \lambda_2 & * & & \vdots \\
\vdots & & & & \ddots & * & \ddots & \vdots \\
\vdots & & & & & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & *
\end{pmatrix}.
$$

Then we can find a unitary transform $\mathbf{H}$ such that

$$
\mathbf{H}^H\mathbf{R}\mathbf{H} = \begin{pmatrix}
* & * & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & \ddots & \ddots & & & & & \vdots \\
\vdots & \ddots & * & * & & & & \vdots \\
\vdots & & \ddots & \lambda_2 & \beta & & & \vdots \\
\vdots & & & \ddots & \lambda_1 & * & & \vdots \\
\vdots & & & & \ddots & * & \ddots & \vdots \\
\vdots & & & & & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & *
\end{pmatrix},
$$

where $*$ denotes elements that are not pertinent to the argument.

Exercise 215 Prove this using Lemma 2.

The rest of the proof follows now by using this observation repeatedly in a *bubble-sort* like operation to move the diagonal entries of $\mathbf{R}$ into the right positions.

Exercise 216 Provide the details. □

This extended version of the Schur decomposition is usually refined even further to facilitate theoretical arguments. In particular we would like to make $\mathbf{R}$ as *diagonal* as possible. Unfortunately, just using a single unitary transformation, the Schur decomposition is the best we can do.

Lemma 4 Let

$$
\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{B} \\ \mathbf{0} & \mathbf{R}_2 \end{pmatrix},
$$

where $\mathbf{R}_i = \lambda_i\mathbf{I} +$ strictly upper triangular matrix, and $\lambda_1 \neq \lambda_2$. Then there exists a non-singular matrix $\mathbf{V}$ such that

$$\mathbf{R} = \mathbf{V} \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{pmatrix} \mathbf{V}^{-1}.$$

Proof.

Exercise 217 Show that there exists a unique solution $\mathbf{X}$, to the system of equations

$$\mathbf{R}_1\mathbf{X} - \mathbf{X}\mathbf{R}_2 + \mathbf{B} = \mathbf{0}.$$

Exercise 218 Show that there exists a unique solution $\mathbf{X}$ to the equation

$$\begin{pmatrix} \mathbf{I} & -\mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 & \mathbf{B} \\ \mathbf{0} & \mathbf{R}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{pmatrix}.$$

Exercise 219 Finish the proof of the lemma. □

Block diagonal We will use the following notation for block diagonal matrices **91**

$$\mathbf{diag}\{\mathbf{R}_i\}_{i=1}^n = \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R}_n \end{pmatrix}.$$

Lemma 5 For every square matrix $\mathbf{A}$ there exists a non-singular matrix $\mathbf{V}$ such that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{diag}\{\mathbf{R}_i\}_{i=1}^M,$$

where $\mathbf{R}_i = \lambda_i \mathbf{I} + \widetilde{\mathbf{R}}_i$, $\widetilde{\mathbf{R}}_i$ are strictly upper triangular matrices, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Proof. Use Lemma 4 repeatedly.

Exercise 220 Fill in the details of the proof. □

The question is can we pick the non-singular matrix $\mathbf{V}$ in the above lemma so as to make $\mathbf{R}_i$ a true diagonal matrix? The answer, unfortunately, is *no*. However, we can come pretty close: we can make it a bi-diagonal matrix with only zeros and ones on the super-diagonal.

Jordan block A Jordan block is a matrix of the form $\lambda\mathbf{I}_n + \mathbf{Z}_n$, where $\mathbf{Z}_n$ is the $n \times n$ *shift up* matrix **92**

$$\mathbf{Z}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Nilpotent A matrix $\mathbf{A}$ is said to be nilpotent is there is a finite integer k for which $\mathbf{A}^k = \mathbf{0}$. **93**

Exercise 221 Show that $\mathbf{Z}_n^n = \mathbf{0}$, and hence nilpotent.

Jordan decomposition Let $\mathbf{R}$ be a nilpotent matrix. Then there exists a non-singular matrix $\mathbf{V}$ such that **94**

$$\mathbf{R} = \mathbf{V} \left(\mathbf{diag}\{\mathbf{Z}_{n_i}\}_{i=1}^{M} \right) \mathbf{V}^{-1}.$$

Proof. Let p be the smallest integer such that $\mathbf{R}^p = \mathbf{0}$. If $p = 0$ we are done. (Why?) So assume $p > 1$. Clearly there exists a $\mathbf{w}$ such that $\mathbf{R}^{p-1}\mathbf{w} \neq \mathbf{0}$. Form the **right Jordan chain**

$$\mathbf{w},\ \mathbf{Rw},\ \mathbf{R}^2\mathbf{w},\ \cdots,\ \mathbf{R}^{p-1}\mathbf{w},$$

and stick them into the matrix

$$\mathbf{W} = \begin{pmatrix} \mathbf{R}^{p-1}\mathbf{w} & \mathbf{R}^{p-2}\mathbf{w} & \cdots & \mathbf{Rw} & \mathbf{w} \end{pmatrix}.$$

Exercise 222 Show that

$$\mathbf{RW} = \mathbf{WZ}_p. \tag{5.1}$$

We claim that $\mathbf{W}$ has full column-rank. To see this consider $\mathbf{Wx} = \mathbf{0}$.

Exercise 223 Multiplying this equation by $\mathbf{R}^{p-1}$ we get $\mathbf{R}^{p-1}\mathbf{Wx} = \mathbf{0}$. From this equation infer that $\mathbf{x}_p = 0$.

Exercise 224 Next multiply by $\mathbf{R}^{p-2}$ to obtain $\mathbf{R}^{p-2}\mathbf{Wx} = \mathbf{0}$ and infer that $\mathbf{x}_{p-1} = 0$.

Exercise 225 Proceed to establish that $\mathbf{x} = \mathbf{0}$ and hence that $\mathbf{W}$ has full column-rank.

Next we construct the *matching* **left Jordan chain**. To do so we first find a vector $\mathbf{y}$ such that

$$\mathbf{y}^H\mathbf{W} = \mathbf{e}_1^H,$$

where $\mathbf{e}_i$ is column i of the identity matrix.

Exercise 226 Why is this possible?

Now form the left Jordan chain

$$\mathbf{y}^H,\ \mathbf{y}^H\mathbf{R},\ \cdots\ \mathbf{y}^H\mathbf{R}^{p-1},$$

and stick them into the matrix

$$\mathbf{Y}^H = \begin{pmatrix} \mathbf{y}^H \\ \mathbf{y}^H\mathbf{R} \\ \vdots \\ \mathbf{y}^H\mathbf{R}^{p-1} \end{pmatrix}.$$

Exercise 227 Show that

$$\mathbf{Y}^H\mathbf{W} = \mathbf{I}.$$

This also establishes that $\mathbf{Y}$ has full column-rank.

Exercise 228 Why? Another way is to imitate the corresponding proof for $\mathbf{W}$.

Exercise 229 Show that

$$\mathbf{Y}^H\mathbf{R} = \mathbf{Z}_p\mathbf{Y}^H. \tag{5.2}$$

Next we find a non-singular matrix $\mathbf{G}$ such that

$$\mathbf{G}^{-1}\mathbf{W} = \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \mathbf{G}^H\mathbf{Y} = \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}.$$

There are many ways to construct G. We do it in two stages.

Exercise 230 Use the SVD of $\mathbf{W}$ to find a non-singular matrix $\mathbf{F}$ such that

$$\mathbf{F}^{-1}\mathbf{W} = \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}.$$

Hint: Make a small modification to the construction of $\mathbf{W}^\dagger$ (which is not invertible).

Since $\mathbf{Y}^H\mathbf{F}\mathbf{F}^{-1}\mathbf{W} = \mathbf{I}$, it follows that

$$\mathbf{Y}^H\mathbf{F} = \begin{pmatrix} \mathbf{I} & \mathbf{Y}_2^H \end{pmatrix}.$$

Exercise 231 Prove it.

Now observe that block Gaussian elimination

$$\begin{pmatrix} \mathbf{I} & \mathbf{Y}_2^H \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{Y}_2^H \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{Y}_2^H \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} = \mathbf{I},$$

provides the necessary correction and we obtain

$$\mathbf{G} = \mathbf{F} \begin{pmatrix} \mathbf{I} & -\mathbf{Y}_2^H \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Exercise 232 Verify that this $\mathbf{G}$ does indeed satisfy all the desired properties.

Using this $\mathbf{G}$ we convert Equations 5.1 and 5.2 into

$$\left(\mathbf{G}^{-1}\mathbf{R}\mathbf{G}\right)\begin{pmatrix}\mathbf{I}\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{Z}_p\\ \mathbf{0}\end{pmatrix}$$
$$\begin{pmatrix}\mathbf{I} & \mathbf{0}\end{pmatrix}\left(\mathbf{G}^{-1}\mathbf{R}\mathbf{G}\right) = \begin{pmatrix}\mathbf{Z}_p & \mathbf{0}\end{pmatrix}.$$

Exercise 233 Verify these formulas.

From this we can verify that $\mathbf{G}^{-1}\mathbf{R}\mathbf{G}$ is a 2×2 block diagonal matrix, with the $(1, 1)$-block being $\mathbf{Z}_p$. Now we can proceed by induction to handle the 2×2 block.

Exercise 234 Complete the proof. □

To summarize the final Jordan decomposition theorem says that for every square matrix $\mathbf{A}$ there exists a non-singular matrix $\mathbf{V}$ such that $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is a block diagonal matrix where each block is of the form

$$\lambda\mathbf{I} + \mathbf{diag}\{\mathbf{Z}_{n_i}\}_{i=1}^{M},$$

where λ, n_i and M can vary from block to block.

5.2 Invariant subspaces

Jordan chains made a magical appearance in the proof. A good way to see how they arise is to consider the uniqueness of the decomposition.

Eigenvalue A complex number λ such that $\mathbf{A} - \lambda\mathbf{I}$ is singular is called an eigenvalue of $\mathbf{A}$. **95**

Eigenvector A non-zero column vector $\mathbf{v}$ is said to be an eigenvector associated with the eigenvalue λ of the matrix $\mathbf{A}$ if $\mathbf{Av} = \lambda\mathbf{v}$. **96**

Invariant subspace A subspace $\mathcal{V}$ is said to be an invariant subspace of the matrix $\mathbf{A}$ if for every $\mathbf{v} \in \mathcal{V}$ we have $\mathbf{Av} \in \mathcal{V}$. **97**

Similarity transformation A matrix $\mathbf{A}$ is said to be similar to a matrix $\mathbf{B}$ if there exists a non-singular matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{VBV}^{-1}$. We also say that $\mathbf{A}$ and $\mathbf{B}$ are related by a similarity transformation. **98**

Exercise 235 Show that if λ is an eigenvalue of $\mathbf{A}$ then it is also an eigenvalue of $\mathbf{VAV}^{-1}$.

Exercise 236 Show that λ is an eigenvalue of the upper triangular matrix $\mathbf{R}$ iff λ is one of the diagonal entries of $\mathbf{R}$.

Lemma 6 The eigenvalues of a matrix $\mathbf{A}$ are exactly the numbers that arise on the diagonal of the upper-triangular matrix $\mathbf{R}$ in the Schur decomposition of $\mathbf{A}$.

Exercise 237 Show that the **trace** of two similar matrices are equal.

Example 4 Consider the matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{pmatrix}.$$

It is clear that the eigenvalues can only be the numbers 1 and 2.

But is the above matrix similar to

$$\mathbf{S} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}?$$

Exercise 238 Show that the two matrices defined above, $\mathbf{R}$ and $\mathbf{S}$, are not similar to each other.

This raises the question of uniqueness of the eigenvalues. It is clear that the distinct numbers that comprise the eigenvalues of a matrix are unique. (Why?) But, what is not clear is if their multiplicities as they occur on the diagonal of the upper-triangular matrix in the Schur decomposition are unique. The above example seems to suggest that they must be unique, and we will proceed to establish it. The idea is to show that the multiplicity of an eigenvalue has a unique geometrical interpretation. We will actually show much more. We will show that the number and size of the Jordan blocks asscoiated with the unique eigenvalue λ are also unique.

For the rest of this section let $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$ denote a Jordan decomposition of the matrix $\mathbf{A}$. Furthermore let λ_i for $i = 1, \ldots, N$, denote the *distinct* eigenvalues of $\mathbf{A}$. Note that the λ_i's are unique by our previous arguments. It is clear that $\mathbf{dim}(\mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I})) = M_{i;1}$ is a well-defined positive number.

Exercise 239 Show that $M_{i;1}$ denotes the number of Jordan blocks of size greater than or equal to one with eigenvalue λ_i. *Hint*: $\mathbf{J}$ is upper triangular and $\mathbf{J} - \lambda_i \mathbf{I}$ has some nilpotent diagonal blocks, which are the only ones that matter in this calculation.

It follows that the number of Jordan blocks asscoiated with the eigenvalue λ is a unique fixed number. Note, this does not imply (right now) that the multiplicity of λ is unique.

Now define $M_{i;2} = \mathbf{dim}(\mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I})^2)$. Again, $M_{i;2}$ is a well-defined unique positive number.

Exercise 240 Show that $\mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I}) \subseteq \mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I})^2$ and hence $M_{i;2} \geq M_{i;1}$.

Exercise 241 Show that $M_{i;2} - M_{i;1}$ is the number of Jordan blocks associated with the eigenvalue λ_i that are of size greater than or equal to two. To do this compute a basis for $\mathcal{N}(\mathbf{J} - \lambda_i \mathbf{I})$ and a basis for $\mathcal{N}(\mathbf{J} - \lambda_i \mathbf{I})^2$. Note that a basis for the latter subspace can be obtained by extending the basis for for the former subspace with a few well-chosen vectors that are associated with the null-vectors of Jordan blocks of size greater than 1.

Exercise 242 Conclude that the number of Jordan blocks of size 1 associated with the eigenvalue λ_i is exactly $2M_{i;1} - M_{i;2}$, which is a unique well-defined non-negative number.

We now rinse and repeat to show that the blocks of bigger sizes must also be unique. Let $M_{i;3} = \mathbf{dim}(\mathcal{N}(\mathbf{A} - \lambda_i \mathbf{I})^3)$.

Exercise 243 Show that $M_{i;3} - M_{i;2}$ is the number of Jordan blocks of size greater than or equal to 3 that are associated with the eigenvalue λ_i.

Clearly we can keep this up and prove that the number and size of each Jordan block is unique and well-defined for a given matrix.

Exercise 244 Make sure that you understand clearly what is going on.

This only leaves the question of the uniqueness of matrix $\mathbf{V}$ in the Jordan decomposition. Unfortunately the matrix is not fully unique. For example, the position of the Jordan blocks inside $\mathbf{J}$ is not unique, thereby implying that the nmatrix $\mathbf{V}$ itself is not unique. However, the columns of $\mathbf{V}$ and the rows of $\mathbf{V}^{-1}$ describe (are bases for) certain invariant subspaces of $\mathbf{A}$, and these invariant subspaces are unique. The previous proof illustrates this point and we say no more about it.

5.3 Difference Equations

So what can we do with spectral decompositions that we could not do with the SVD? We have already seen examples, like the Stein equation, which can be more efficiently solved via spectral decompositions. However the classical examples are infinite sets of equations where spectral decompositions (for now at least) are the only way.

Let $\mathbf{u}[n] \in \mathbb{C}^N$ for $n = 0, 1, 2, \ldots$, be a sequence of unknown column vectors that satisfy the constraints

$$\mathbf{u}[n+1] = \mathbf{A}\mathbf{u}[n] + \mathbf{f}[n], \tag{5.3}$$

where $\mathbf{A} \in \mathbb{C}^{N\times N}$ and $\mathbf{f}[n] \in \mathbb{C}^N$ and are both known quantities. The question is to find all sequences $\mathbf{u}[n]$ that satisfy the above constraints.

Exercise 245 Write the above set of equations in the form $\mathbf{F}\mathbf{x} = \mathbf{b}$.

Note that there are an infinite number of unknowns and equations. So, even though the constraints are linear equations it is not easy to develop a procedure like Gaussian elimination to find the solutions. Fortunately it turns out that a spectral decomposition of $\mathbf{A}$ is sufficient.

The idea is to first figure out the nullspace of the associated matrix. Consider the so-called **homogenous equations**

$$\mathbf{u}_h[n+1] = \mathbf{A}\mathbf{u}_h[n], \qquad n \geq 0.$$

It is clear that the only solutions are of the form

$$\mathbf{u}_h[n] = \mathbf{A}^n\mathbf{u}_h[0].$$

From this we can guess that a solution of the equations is

$$\mathbf{u}_p[n+1] = \sum_{k=0}^{n} \mathbf{A}^{n-k}\mathbf{f}[k],$$

assuming $\mathbf{u}_p[0] = \mathbf{0}$.

Exercise 246 Verify that $\mathbf{u}_p$ does indeed satisfy the difference equation 5.3.

Therefore the general solution is

$$\mathbf{u}[n] = \mathbf{A}^n\mathbf{u}[0] + \mathbf{A}^{n-1}\mathbf{f}[0] + \mathbf{A}^{n-2}\mathbf{f}[1] + \cdots + \mathbf{A}^0\mathbf{f}[n-1].$$

Exercise 247 Verify this.

This formula is a bit cumbersome to use. A simplification is available via the Jordan decomposition $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$.

Exercise 248 Show that $\mathbf{A}^n = \mathbf{V}\mathbf{J}^n\mathbf{V}^{-1}$.

Remember that $\mathbf{J}$ is block diagonal with each diagonal block of the form $\lambda\mathbf{I} + \mathbf{Z}_p$. Therefore we only need to figure out a formula for $(\lambda\mathbf{I} + \mathbf{Z}_p)^n$. (Why?)

Exercise 249 Prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

for $a, b \in \mathbb{C}$.

Exercise 250 Show that if $\mathbf{AB} = \mathbf{BA}$ then

$$(\mathbf{A}+\mathbf{B})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k \mathbf{B}^{n-k}.$$

Exercise 251 Show that $(\lambda\mathbf{I} + \mathbf{Z}_p)^n$ is an upper triangular matrix with

$$\frac{n!}{(n-k)!k!}\lambda^{n-k}$$

as the entry in the k-th super-diagonal. So λ^n is the entry on the main diagonal, for example.

Exercise 252 Using the Jordan decomposition develop a simple formula for $\mathbf{V}^{-1}\mathbf{u}[n]$, the solution of the difference equation in terms of $\mathbf{V}^{-1}\mathbf{f}$.

5.4 Matrix-valued functions

We now define differentiation and integration of matrix-valued functions. Let $\mathbf{A} : \mathbb{C} \to \mathbb{C}^{m\times n}$, denote a matrix-valued function of a single complex variable. This is usually denoted as $\mathbf{A}(z)$. We define $\frac{d}{dz}\mathbf{A}(z)$ to be an $m \times n$ matrix whose (i, j) entry is the derivative of the (i, j) entry of $\mathbf{A}(z)$. In other words we define differentiation component-wise. Sometimes we will use a super-script prime to denote differentiation: $\mathbf{A}'(z)$.

In a similar manner we define $\int_\Gamma \mathbf{A}(z)dz$ to be an $m \times n$ matrix with the (i, j) component being the corresponding integral of the (i, j) component of $\mathbf{A}(z)$. Note that both differentiation and integration are defined here for matrices of arbitrary size of a single (potentially complex) variable.

Exercise 253 Show that

$$\frac{d}{dt}(\mathbf{A}(t) + \mathbf{B}(t)) = \frac{d}{dt}\mathbf{A}(t) + \frac{d}{dt}\mathbf{B}(t),$$
$$\frac{d}{dt}(\mathbf{A}(t)\mathbf{B}(t)) = \left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{B}(t) + \mathbf{A}\frac{d}{dt}\mathbf{B}(t).$$

Exercise 254 Show that

$$\frac{d}{dt}\mathbf{A}^{-1}(t) = -\mathbf{A}^{-1}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{A}^{-1}(t).$$

Hint: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Exercise 255 Show that

$$\int \mathbf{A}\,\mathbf{B}(t)\,\mathbf{C}\,dt = \mathbf{A}\int \mathbf{B}(t)\,dt\,\mathbf{C},$$

when $\mathbf{A}$ and $\mathbf{C}$ are constant matrices.

A matrix-valued function $\mathbf{A}(t)$ is said to be continuous function of t if each component $\mathbf{A}_{ij}(t)$ is a continuous function of t. Suitable changes should be made for "continuous at a point" and "continuous on a set".

Exercise 256 Let $\mathbf{A}(t)$ be a continuously differentiable matrix-valued function on $[0, 1]$. Show that

$$\int_0^1 \frac{d}{dt}\mathbf{A}(t)\,dt = \mathbf{A}(1) - \mathbf{A}(0).$$

Exercise 257 Let $\mathbf{A}(t)$ be a continous matrix-valued function on the interval $[0, 1]$. Show that

$$\left\| \int_0^1 \mathbf{A}(t)\, dt \right\| \leq \int_0^1 \|\mathbf{A}(t)\|\ dt.$$

Hint: Use Riemann sums to approximate both sides and use the triangle inequality satisfied by norms.

5.5 Functions of matrices

While it is possible to give more examples of infinite sets of equations whose solution is made accessible via spectral decompositions, we will take a more general point of view in this section.

In Section 5.3 we saw the need to understand the internal structure of sums of powers of matrices. In this section we place that in a larger context. Given an analytic functions (like z^n) how to evaluate that function at a given matrix $\mathbf{A}$?

First we need some additional facts from complex analysis. See Section 3.6 for some preliminary facts. Once more, for the next three exercises, *engineering proofs* are good enough. Anything better requires substantially more machinery.

Exercise 258 Extend Exercise 130 to show that if $\Gamma \subset \Omega$ is some simple (not self-intersecting) smooth closed curve in the open set Ω in the complex plane, and f is analytic in Ω then $\int_\Gamma f(z)dz = 0$. *Hint*: Use the fact that $f(z) = F'(z)$ for some suitable analytic function F. Can you suggest a candidate for F?

Exercise 259 Extend Cauchy's formula (Exercise 131) to the case where the contour of integration $\Gamma \subset \Omega$ is not necessarily a circle, but just a simple smooth closed curve:

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz.$$

Hint: Starting with the circle deform it to the desired curve in pieces using the previous exercise.

Exercise 260 Show that

$$\frac{d^n}{da^n} f(a) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-a)^{n+1}} dz.$$

$f(\mathbf{A})$ Let $\mathbf{A}$ be a square matrix. Let f be an analytic function in the open set Ω. Let Γ **99**
be a smooth closed curve in Ω. Suppose all the eigenvalues of $\mathbf{A}$ lie inside the open set bounded by Γ. Then we define

$$f(\mathbf{A}) = \frac{1}{2\pi i} \int_\Gamma f(z)(z\mathbf{I} - \mathbf{A})^{-1} dz.$$

Implicit in this definition is that the integral is well-defined and that the choice of the curve Γ is immaterial as long as it is simple, lies inside Ω and encloses in its strict interior all the eigenvalues of $\mathbf{A}$.

Let $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$ denote the Jordan decomposition of $\mathbf{A}$. Show that

$$\mathbf{V}^{-1}f(\mathbf{A})\mathbf{V} = \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{I}-\mathbf{J})^{-1}dz.$$

Therefore it is enough to verify these assertions when $\mathbf{A}$ is a simple Jordan block. (Why?)

Let $\mathbf{J}_p(\lambda) = \lambda\mathbf{I} + \mathbf{Z}_p$.

Exercise 261 Show that

$$(z\mathbf{I}-\mathbf{J}_p(\lambda))^{-1} = \begin{pmatrix} \frac{1}{z-\lambda} & \frac{1}{(z-\lambda)^2} & \frac{1}{(z-\lambda)^3} & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

which is an upper-triangular Toeplitz matrix.

Exercise 262 Show that

$$\frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{I}-\mathbf{J}_p(\lambda))^{-1}dz = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

This clearly shows the independence of the definition of $f(\mathbf{A})$ on the curve Γ.

The Cauchy integral formula has a certain advantage for defining functions of matrices: it is global. However Taylor series work better sometimes.

Let $f(z) = \sum_{n=0}^\infty c_n(z-c)^n$ for $|z-c| < R$. Let all the eigenvalues of $\mathbf{A}$ lie inside the circle $\Omega = |z-c| < R$. Let Γ denote a simple closed curve inside Ω. Then for any a inside the interior of Γ it is clear that

$$f(a) = \sum_{n=0}^\infty c_n(a-c)^n = \frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}dz.$$

This suggests that $f(\mathbf{A}) = \sum_{n=0}^\infty c_n(\mathbf{A}-c\mathbf{I})^n$ should be true.

Exercise 263 Show that $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(z-c)^n$, where $f^{(n)}$ denotes the n-th order derivative of f.

Exercise 264 Let $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$ denote the Jordan decompoition of $\mathbf{A}$. Show that $\sum_{n=0}^\infty c_n(\mathbf{A} - c\mathbf{I})^n = \mathbf{V}(\sum_{n=0}^\infty c_n(\mathbf{J}-c\mathbf{I})^n)\mathbf{V}^{-1}$.

Therefore it is sufficient to check if the Talyor series can be used to evaluate $f(\mathbf{A})$ when all the eigenvalues of $\mathbf{A}$ lie inside the circle of convergence.

Exercise 265 Show that

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (\mathbf{J}_p(\lambda) - c\mathbf{I})^n = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

This show that the Taylor series expansion can be used to evaluate $f(\mathbf{A})$ but only when the eigenvalues lie inside the circle of convergence.

Example 5 Let $f(z) = \sqrt{z}$. Unfortunately $\sqrt{z}$ is multi-valued and we must specify a branch to use. Let $z = re^{i\theta}$ denote the polar decomposition of the complex number z with $-\pi < \theta \leq \pi$. Pick the branch for the square-root such that $f(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$; that is, $f(z)$ lies in the right-half plane. Note that $f(z)$ is discontiuous across the negative real line. Therefore the negative real line is called the **branch cut** for $f(z)$. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the eigenvalues of $\mathbf{A}$ are 1 and -1. Clearly the eigenvalues of $\mathbf{A}$ do not lie in an open set Ω in which $f(z)$ is analytic. Therefore neither Cauchy's formula nor Taylor series expansions can be used to evaluate $f(\mathbf{A})$ in this case. However, if we just want to solve the equation $\mathbf{B}^2 = \mathbf{A}$, then it is easy to write down several solutions

$$\mathbf{B} = \begin{pmatrix} f_1(1) & 0 \\ 0 & f_2(-1) \end{pmatrix},$$

where f_1 and f_2 can be two different branches of the square root function. This corresponds to picking the branch cut in such a way as to avoid all the eigenvalues of $\mathbf{A}$ and allowing them to lie in a single connected open region.

Entire functions, functions that are analytic in the entire complex plane, do not suffer from this problem. Both Cauchy's formula and Taylor series expansions will *always* work. The most common examples of entire functions are the exponential, sine and cosine.

Another exmaple of a multi-valued function is the logarithm. Again, depending on the location of the eigenvalues either Taylor series (less often), or Cauchy's integral formula (more often), can be used. If both fail to be applicable then the branch cut must be adjusted suitably.

5.6 Differential equations

Let $\mathbf{u}(t)$ be a vector-valued function of the real variable t. Our objective is to find $\mathbf{u}(t)$ that satisfies the differential equation

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{b}(t), \qquad t > 0, \tag{5.4}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{b}(t)$ is a known vector-valued function.

First some auxiliary facts.

Exercise 266 Suppose $t\mathbf{A}$ has all its eigenvalues inside Ω where f is analytic. Show that

$$\frac{d}{dt}f(t\mathbf{A}) = f'(t\mathbf{A})\mathbf{A} = \mathbf{A}f'(t\mathbf{A}).$$

The proof is quite easy if you use a Taylor series expansion, but not general enough. In general you have to use Cauchy's formula and the fact that since the integral is absolutely converging you can differentiate inside the integral.

We first look at the homogenous equation

$$\frac{d}{dt}\mathbf{u}_h(t) = \mathbf{A}\mathbf{u}_h(t), \qquad t > 0.$$

Exercise 267 Verify that a solution is $\mathbf{u}_h(t) = e^{t\mathbf{A}}\mathbf{u}_h(0)$.

With a little effort one can establish that this is the only solution. One approach is to use the Jordan decomposition to reduce the problem to a set of single variable ODEs and appeal to the scalar theory. Here we take an approach via Picard iteration that also generalizes to non-constant coefficient ODEs.

Let $[0, T]$ be the interval over which a solution to the ODE

$$\frac{d}{dt}\mathbf{u}_u(t) = \mathbf{A}\mathbf{u}_u(t), \qquad \mathbf{u}_u(0) = 0,$$

exists. If we can show that $\mathbf{u}_u(t) = 0$ then we would have established uniqueness. (Why?). Since the derivative of $\mathbf{u}_u$ exists, it must be continuous. Let $\|\mathbf{u}_u(t)\| \leq L < \infty$ for $t \in [0, T]$.

Exercise 268 Show that

$$\mathbf{u}_u(t) = \int_0^t \mathbf{A}\mathbf{u}_u(s)ds.$$

Exercise 269 Show that

$$\|\mathbf{u}_u(t)\| \leq t\|\mathbf{A}\|L.$$

Hint: See Exercise 257.

Exercise 270 Repeat the above argument and show that

$$\|\mathbf{u}_u(t)\| \leq \frac{t^n\|\mathbf{A}\|^n}{n!}L, \quad n \geq 1.$$

Exercise 271 Conclude that $\mathbf{u}_u(t) = 0$ for $t \in [0, T]$.

Now that we have uniqueness, we can look at the form of the homogenous solution and guess that a particular solution of the differential equation is

$$\mathbf{u}_p(t) = \int_0^t e^{(t-s)\mathbf{A}}\mathbf{b}(s)ds,$$

assuming that $\mathbf{u}_p(0) = \mathbf{0}$.

Exercise 272 Show that $e^{(t+s)\mathbf{A}} = e^{t\mathbf{A}}e^{s\mathbf{A}} = e^{s\mathbf{A}}e^{t\mathbf{A}}$. Since the exponential is an entire function an easy proof is via a Taylor series expansion for the exponential function.

Exercise 273 Show that $e^{\mathbf{0}} = \mathbf{I}$.

Exercise 274 Show that $e^{-\mathbf{A}} = (e^{\mathbf{A}})^{-1}$.

Exercise 275 Verify that $\mathbf{u}_p(t)$ is indeed a solution of equation 5.4.

Therefore the general solution to equation 5.4 is

$$\mathbf{u}(t) = \int_0^t e^{(t-s)\mathbf{A}}\mathbf{b}(s)ds + e^{t\mathbf{A}}\mathbf{u}(0).$$

5.7 Localization of eigenvalues

One of the most important questions is how does $f(\mathbf{A})$ change when we perturb $\mathbf{A}$. We already considered this question when $f(x) = x^{-1}$ in Section 3.7. An obvious idea is to use the Jordan decomposition to help make this estimate. For example

$$\|f(\mathbf{A})\| = \|f(\mathbf{V}\mathbf{J}\mathbf{V}^{-1})\| \leq \|\mathbf{V}\|\|\mathbf{V}^{-1}\|\|f(\mathbf{J})\|.$$

But this upper bound can be wildly inaccuarate if $\kappa(\mathbf{V}) = \|\mathbf{V}\|\|\mathbf{V}^{-1}\|$ is very large. However, better general-purpose estimates are hard to come by. So one approach is to look for special classes of matrices for which $\kappa(\mathbf{V})$ is small in a suitable norm.

Let $\kappa_2(\mathbf{V}) = \|\mathbf{V}\|_2\|\mathbf{V}^{-1}\|_2$ denote the 2-norm condition number of the matrix $\mathbf{V}$.

Exercise 276 Show that $\kappa_2(\mathbf{V}) \geq 1$. *Hint*: Use the SVD.

Exercise 277 Show that if $\mathbf{V}$ is a unitary matrix then $\kappa_2(\mathbf{V}) = 1$.

Normal matrix A matrix $\mathbf{A}$ is said to be normal if $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A}$. **100**

Exercise 278 Show that unitary and orthogonal matrices are normal.

Symmetry A matrix $\mathbf{A}$ is said to be **101**

- **symmetric** if $\mathbf{A}^T = \mathbf{A}$
- **skew-symmetric** if $\mathbf{A}^T = -\mathbf{A}$
- **Hermitian** if $\mathbf{A}^H = \mathbf{A}$
- **skew-Hermitian** if $\mathbf{A}^H = -\mathbf{A}$

Exercise 279 Show that Hermitian and skew-Hermitian matrices are normal.

Exercise 280 Show that every matrix can be written uniquely as the sum of a Hermitian and a skew-Hermitian matrix. *Hint*:

$$\mathbf{A} = \frac{\mathbf{A} + \mathbf{A}^H}{2} + \frac{\mathbf{A} - \mathbf{A}^H}{2}.$$

Theorem 2 Let $\mathbf{A}$ be normal. Then there exists a unitary matrix $\mathbf{Q}$ and a diagonal matrix Λ such that $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^H$.

In other words for normal matrices the Schur decomposition is also the Jordan decomposition with each Jordan block being of size one. Furthermore there is a full set of orthonormal eigenvectors.

Proof. The proof follows from the following fact.

Lemma 7 If $\mathbf{R}$ is an upper triangular normal matrix then it is diagonal.

Exercise 281 Prove the lemma. *Hint*: Write $\mathbf{R}$ as a 2×2 block matrix and solve the four resulting equations.

Exercise 282 Prove the theorem. □

It follows that for normal matrices $\|f(\mathbf{A})\| = \|f(\Lambda)\|$, where the diagonal entries of Λ are the eigenvalues of $\mathbf{A}$. Therefore it becomes essential to locate, at least approximately, the eigenvalues of $\mathbf{A}$ in the complex plane.

Let $\mathbf{A}$ be normal with Schur decomposition $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^H$. Consider the expression $(\mathbf{Q}^H\mathbf{A}\mathbf{Q})^H$ and use it to prove the next three exercises.

Exercise 283 Show that the eigenvalues of a unitary matrix must lie on the unit circle.

Exercise 284 Show that the eigenvalues of a Hermitian matrix must be real.

Exercise 285 Show that the eigenvalues of a skew-Hermitian matrix must be purely imaginary.

Exercise 286 Show that the eigenvectors of a normal matrix corresponding to distinct eigenvalues must be mutually orthogonal. *Hint*: Use the Schur decomposition.

Exercise 287 Write down a family of normal matrices that is neither unitary nor Hermitian nor skew-Hermitian. *Hint*: Use the Schur decomposition.

Exercise 288 Show that $e^{\text{skew-Hermitian}} = \text{unitary}$.

Exercise 289 Show that $e^{i\,\text{Hermitian}} = \text{unitary}$.

Exercise 290 Let $\mathbf{A}$ be a square real matrix. Suppose λ is an eigenvalue of $\mathbf{A}$ with a non-zero imaginary part.

- Show that the corresponding eigenvector $\mathbf{v}$, must have real and imaginary parts that are linearly independent when considered as real vectors..
- Show that $\bar{\lambda}$ must also be an eigenvalue of $\mathbf{A}$.
- Show that an eigenvector for $\bar{\lambda}$ can be constructed from $\mathbf{v}$.

Exercise 291 Show that a real orthogonal matrix with an odd number of rows and columns must have either 1 or -1 as one of its eigenvalues.

5.8 Real symmetric matrices

Real symmetric matrices play the role of real numbers in matrix analysis.

Exercise 292 Let $\mathbf{A} = \mathbf{A}_R + i\mathbf{A}_I$ denote the real and imaginary parts of the $m \times n$ matrix $\mathbf{A}$. Show that

$$T(\mathbf{A}) = \begin{pmatrix} \mathbf{A}_R & \mathbf{A}_I \\ -\mathbf{A}_I & \mathbf{A}_R \end{pmatrix},$$

is a faithful representation of the complex matrix $\mathbf{A}$ as a real matrix of twice the size, in the sense that for all complex matrices $\mathbf{A}$ and $\mathbf{B}$

- $T(\alpha\mathbf{A}) = \alpha T(\mathbf{A})$
- $T(\mathbf{A}^H) = T(\mathbf{A})^T$
- $T(\mathbf{A} + \mathbf{B}) = T(\mathbf{A}) + T(\mathbf{B})$
- $T(\mathbf{AB}) = T(\mathbf{A})T(\mathbf{B})$

whenever the operations are well-defined.

Exercise 293 Show that

- $T(\text{unitary}) = \text{orthogonal}$
- $T(\text{Hermitian}) = \text{symmetric}$
- $T(\text{skew-Hermitian}) = \text{skew-symmetric}$

Theorem 3 Let A be a real symmetric matrix. Then there exists a real orthogonal matrix $\mathbf{Q}$ and a real diagonal matrix Λ such that $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ and $\Lambda_{i,i} \geq \Lambda_{i+1,i+1}$.

Proof. Just repeat the proof of the Schur decomposition and observe that you can use orthogonal transforms instead of unitary transforms since the eigenvalues are known to be real. Also, symmetry will help to directly produce a diagonal rather than upper-triangular matrix.

Exercise 294 Work out a detailed proof. □

From now on we will use the notation $\Lambda_{ii} = \lambda_i$ for convenience.

Exercise 295 Let $\mathbf{A}$ be a real $m \times n$ matrix. Show that

$$\|\mathbf{A}\|_2 = \max_{\mathbf{0} \neq \mathbf{z} \in \mathbb{C}^n} \frac{\|\mathbf{Az}\|_2}{\|\mathbf{z}\|_2} = \max_{\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}.$$

Hint: Exercise 172 might be useful.

Exercise 296 Redo the proof of the SVD and show that if $\mathbf{A}$ is a real (possibly non-square) matrix, then there exist real orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$, with Σ having non-zero entries only on its principal diagonal, and $\Sigma_{i,i} \geq \Sigma_{i+1,i+1} \geq 0$.

Exercise 297 Let $\mathbf{A}$ be a real symmetric matrix.

- Let $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ be its Schur decomposition. Show how to use it to write down the SVD of $\mathbf{A}$.
- Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ be its SVD. Is it always possible to infer the Schur decomposition directly from the SVD? *Hint*:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 298 Let $\mathbf{A}$ be a $m \times n$ matrix. Use the SVD of $\mathbf{A}$ to write down the Schur decomposition of $\mathbf{A}^H\mathbf{A}$ and $\mathbf{A}\mathbf{A}^H$. You *cannot* use these formulas to directly infer the SVD of $\mathbf{A}$ from the Schur decompositions of $\mathbf{A}^H\mathbf{A}$ and $\mathbf{A}\mathbf{A}^H$. Why?

Exercise 299 Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix with eigenvalues λ_i. Show that for real $\mathbf{x} \neq \mathbf{0}$

$$\lambda_n \leq \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_1.$$

Hint: Use the Schur decomposition to convert the **Rayleigh quotient** (the fractional middle term above) into the form

$$\frac{\mathbf{y}^T\Lambda\mathbf{y}}{\mathbf{y}^T\mathbf{y}}.$$

Courant-Fischer Let $\mathbf{A}$ be a real $n \times n$ symmetric matrix with eigenvalues λ_i in decreasing order $\lambda_i \geq \lambda_{i+1}$. Then **102**

$$\lambda_k = \max_{\mathbf{dim}(\mathcal{U})=k} \min_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Proof.

Exercise 300 Use Exercise 299 to prove the theorem for $k = 1$ and $k = n$.

Now fix k to be a number between 1 and n. Let $\mathbf{q}_i$ denote column i of the matrix $\mathbf{Q}$ from the Schur decomposition of $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$. First pick $\mathcal{U} = \mathbf{span}\{\mathbf{q}_1, \cdots, \mathbf{q}_k\}$.

Exercise 301 Show that for this choice of $\mathcal{U}$

$$\min_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_k.$$

Hint: Note that $\mathbf{A}\mathbf{q}_i = \lambda_i\mathbf{q}_i$. Then look at Exercise 299.

It follows that

$$\max_{\mathbf{dim}(\mathcal{U})=k} \min_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \lambda_k.$$

Next let $\mathcal{U}$ be any subspace of $\mathbb{R}^n$ of dimension k. Consider the subspace $\mathcal{V} = \mathbf{span}\{\mathbf{q}_k, \ldots, \mathbf{q}_n\}$. Since $\mathbf{dim}(\mathcal{U}) = k$ and $\mathbf{dim}(\mathcal{V}) = n - k + 1$, it follows that $\mathcal{U} \cap \mathcal{V} \neq \{\mathbf{0}\}$.

Exercise 302 Show that $\mathbf{dim}(\mathcal{U} \cap \mathcal{V}) \geq 1$.

Pick a non-zero $\mathbf{z} \in \mathcal{U} \cap \mathcal{V}$. It can be represented as $\mathbf{z} = \sum_{i=k}^{n} \alpha_i \mathbf{q}_i$.

Exercise 303 Show that

$$\frac{\mathbf{z}^T\mathbf{A}\mathbf{z}}{\mathbf{z}^T\mathbf{z}} \leq \lambda_k.$$

Hint: Use Exercise 299.

From this it follows that for any k-dimensional subspace $\mathcal{U}$ of $\mathbb{R}^n$

$$\min_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_k.$$

Therefore it follows that

$$\max_{\mathbf{dim}(\mathcal{U})=k} \min_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_k.$$

Therefore the theorem is true. □

Exercise 304 Show that

$$\lambda_k = \min_{\mathbf{dim}(\mathcal{U})=n-k+1} \max_{\mathbf{0}\neq\mathbf{x}\in\mathcal{U}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Hint: Consider $-\mathbf{A}$.

We can now derive a perturbation result for the eigenvalues of real symmetric matrices.

Theorem 4 Let $\mathbf{A}$ and $\mathbf{E}$ be real symmetric $n \times n$ matrices. Let $\lambda_i(\mathbf{A})$ denote the eigenvalues of $\mathbf{A}$ in decreasing order. Then

$$\lambda_i(\mathbf{A}) + \lambda_n(\mathbf{E}) \leq \lambda_i(\mathbf{A} + \mathbf{E}) \leq \lambda_i(\mathbf{A}) + \lambda_1(\mathbf{E}).$$

This shows that the eigenvalues of real symmetric matrices depend continuously on the matrix entries as long as the change leaves the matrix real and symmetric. Furthermore it shows that the eigenvalues of a real symmetric matrix are *well-conditioned* with respect to absolute perturbations.

Proof. Let $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ denote the Schur decomposition of $\mathbf{A}$ with eigenvalues in decreasing order. Let $\mathbf{q}_i$ denote column i of $\mathbf{Q}$ and let $\mathcal{U}_k = \mathbf{span}\{\mathbf{q}_k, \ldots, \mathbf{q}_n\}$.

Exercise 305 Using the min-max version of the Courant-Fischer theorem in Exercise 304 to establish that

$$\lambda_k(\mathbf{A} + \mathbf{E}) \leq \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{U}_k} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} + \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{U}_k} \frac{\mathbf{x}^T\mathbf{E}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Exercise 306 From this infer that

$$\lambda_k(\mathbf{A} + \mathbf{E}) \leq \lambda_k(\mathbf{A}) + \lambda_1(\mathbf{E}).$$

Exercise 307 From this infer that

$$\lambda_k(\mathbf{A} + \mathbf{E}) \geq \lambda_k(\mathbf{A}) + \lambda_n(\mathbf{E})$$

Hint: You can use the previous inequality with $\mathbf{A} \to \mathbf{A} + \mathbf{E}$ and $\mathbf{E} \to -\mathbf{E}$, or you can repeat the earlier argument with the max-min version of the Courant-Fischer theorem. □

Exercise 308 Show that $\|\mathbf{A}\|_2 = \max\{|\lambda_1(\mathbf{A})|, |\lambda_n(\mathbf{A})|\}$, when $\mathbf{A}$ is a real $n \times n$ symmetric matrix, with eigenvalues in decreasing order.

Exercise 309 Show that $|\lambda_i(\mathbf{A} + \mathbf{E}) - \lambda_i(\mathbf{A})| \leq \|\mathbf{E}\|_2$, when $\mathbf{A}$ and $\mathbf{E}$ are real symmetric matrices with eigenvalues in decreasing order.

Next we consider perturbations that can change the size of the matrix.

Cauchy Interlacing Theorem Let $\mathbf{A}$ be a real $n \times n$ symmetric matrix partitioned as follows **103**

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{c}^T & \delta \end{pmatrix},$$

where δ is a real number. Then

$$\lambda_n(\mathbf{A}) \le \lambda_{n-1}(\mathbf{B}) \le \cdots \le \lambda_k(\mathbf{B}) \le \lambda_k(\mathbf{A}) \le \lambda_{k-1}(\mathbf{B}) \le \cdots \le \lambda_1(\mathbf{B}) \le \lambda_1(\mathbf{A}).$$

Proof. Let $\mathbf{B} = \mathbf{Q}\Lambda\mathbf{Q}^T$ denotes the Schur decomposition of $\mathbf{B}$ with eigenvalues in decreasing order. Let $\mathbf{q}_i$ denote column i of $\mathbf{Q}$. Define the range space

$$\mathcal{U}_k = \mathcal{R}\begin{pmatrix} \mathbf{q}_{k-1} & \cdots & \mathbf{q}_{n-1} \\ 0 & \cdots & 0 \end{pmatrix}.$$

Note that there are only $n-1$ columns in $\mathbf{Q}$.

Exercise 310 Using the min-max version of the Courant-Fischer theorem show that

$$\lambda_k(\mathbf{A}) \le \max_{\mathbf{0}\ne\mathbf{x}\in\mathcal{U}_k} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_{k-1}(\mathbf{B}).$$

Exercise 311 Either apply the previous inequality to $-\mathbf{A}$ and establish that

$$\lambda_k(\mathbf{B}) \le \lambda_k(\mathbf{A}),$$

or repeat the argument with the max-min version of the Courant-Fischer theorem. □

5.9 Cholesky factorization

While the Schur decomposition reveals a lot about symmetric matrices, it is hard to compute since in general there are no closed-form formulas.

Positive semi-definite A matrix $\mathbf{A}$ is said to be positive semi-definite if $\mathbf{x}^H \mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$. **104**

Exercise 312 Show that if a matrix is Hermitian positive semi-definite then the diagonal entries are non-negative.

Principal sub-matrix A matrix $\mathbf{B}$ is said to be a principal sub-matrix of the matrix $\mathbf{A}$ if there exists a permutation $\mathbf{P}$ such that **105**

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \mathbf{B} & * \\ * & * \end{pmatrix} \mathbf{P}^T.$$

Exercise 313 Show that every principal sub-matrix of a positive semi-definite matrix is positive semi-definite.

Exercise 314 Show that the eigenvalues of a Hermitian positive semi-definite matrix are non-negative.

Exercise 315 Show that if $\mathbf{A}\mathbf{A}^H$ is a Hermitian positive semi-definite matrix.

Exercise 316 Show that every Hermitian positive semi-definite matrix can be written in the form $\mathbf{A}\mathbf{A}^H$ for some suitable $\mathbf{A}$. *Hint*: Use the Schur decomposition.

Positive definite A matrix $\mathbf{A}$ is said to be positive definite if $\mathbf{x}^H \mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. **106**

Exercise 317 Repeat the previous exercises with suitable modifications for Hermitian positive definite matrices.

Cholesky factorization Let $\mathbf{A}$ be a Hermitian positive definite matrix. Then there exists a non-singular lower-triangular matrix $\mathbf{G}$ with positive diagonal entries such that $\mathbf{A} = \mathbf{G}\mathbf{G}^H$. **107**

Proof. The proof is a repetition of the **LU** factorization proof, except that it does not require the use of permutations.

Exercise 318 Furnish the proof. □

5.10 Problems

Problem 11 Let $\mathbf{A}$ be a real (possibly non-square) matrix. Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{pmatrix}.$$

Show that $\mathbf{B}$ is a real symmetric matrix. Show that the Schur decomposition of $\mathbf{B}$ can be written in terms of the SVD of $\mathbf{A}$. *Hint*: You can find a permutation Π such that

$$\Pi \begin{pmatrix} \mathbf{0} & \Sigma^T \\ \Sigma & \mathbf{0} \end{pmatrix} \Pi^T,$$

is a block diagonal matrix with each block of size 2×2 at most.

Problem 12 Let $\mathbf{A}$ and $\mathbf{E}$ be real (possibly non-square) matrices. Let $\sigma_i(\mathbf{A})$ denote the singular values of $\mathbf{A}$ in decreasing order. Show that

$$|\sigma_i(\mathbf{A} + \mathbf{E}) - \sigma_i(\mathbf{A})| \leq \|\mathbf{E}\|_2.$$

Problem 13 Let σ_i denote the singular values of $\mathbf{A}$. Show that σ_{k+1} is the 2-norm distance of $\mathbf{A}$ to the nearest rank-k matrix.

Problem 14 Let $\mathbf{A}$ be an $m \times n$ real matrix partitioned as follows

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} \\ \mathbf{c}^T \end{pmatrix},$$

where $\mathbf{c}$ is a real column vector. Show that

$$\cdots \leq \sigma_k(\mathbf{B}) \leq \sigma_k(\mathbf{A}) \leq \sigma_{k-1}(\mathbf{B}) \leq \cdots \leq \sigma_1(\mathbf{B}) \leq \sigma_1(\mathbf{A})$$

where $\sigma_i(\mathbf{A})$ denotes the singular values of $\mathbf{A}$ in decreasing order.

Problem 15 Use the real and imaginary parts of the SVD of $\mathbf{A}$, to write down the real SVD of the real matrix $T(\mathbf{A})$, where T is defined in Exercise 292.

Problem 16 **Wielandt–Hoffman**. This problem is quite challenging. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ normal matrices. Let $\lambda_i(\mathbf{A})$ denote the eigenvalues of $\mathbf{A}$. Show that

$$\min_{\sigma \in \text{Permutations}} \sum_{i=1}^{n} |\lambda_i(\mathbf{A}) - \lambda_{\sigma(i)}(\mathbf{B})|^2 \leq \|\mathbf{E}\|_F^2.$$

Problem 17 Show that

$$\min_{\mathbf{X} \in \mathbb{C}^{n \times m}} \|\mathbf{A}\mathbf{X} - \mathbf{I}\|_F = \|\mathbf{A}\mathbf{A}^\dagger - \mathbf{I}\|_F.$$

6 Tensor Algebra

In this chapter we consider the case when both entries in $\mathbf{A}$ and $\mathbf{x}$ must be considered as variables in the expression $\mathbf{Ax}$. In general more terms could be involved in the product; so we are concerned with multi-linear analysis.

6.1 Kronecker product

Again we prefer to introduce Kronecker products of matrices as a direct concrete realization of tensor products.

Kronecker product Let $\mathbf{A}$ and $\mathbf{B}$ be two matrives. We define the tensor or Kronecker product as follows **108**

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B} & \mathbf{A}_{12}\mathbf{B} & \cdots \\ \mathbf{A}_{21}\mathbf{B} & \mathbf{A}_{22}\mathbf{B} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Exercise 319 Show that if $\mathbf{x}$ and $\mathbf{y}$ are column vectors then

$$\mathbf{x}\mathbf{y}^H = \mathbf{x} \otimes \mathbf{y}^H = \mathbf{y}^H \otimes \mathbf{x}.$$

Exercise 320 Give an example where $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$.

Exercise 321 Show that there are permutations $\mathbf{P}_1$ and $\mathbf{P}_2$ such that $\mathbf{A} \otimes \mathbf{B} = \mathbf{P}_1(\mathbf{B} \otimes \mathbf{A})\mathbf{P}_2$.

Exercise 322 Show that

- $(\alpha\mathbf{A}) \otimes \mathbf{B} = \alpha(\mathbf{A} \otimes \mathbf{B}) = \mathbf{A} \otimes (\alpha\mathbf{B})$.
- $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$.
- $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$.
- $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$
- $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$
- $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$
- $\mathbf{I} \otimes \mathbf{I} = \mathbf{I}$
- $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$
- Hermitian $\otimes$ Hermitian $=$ Hermitian

- Unitary $\otimes$ Unitary = Unitary
- Hermitian $\otimes$ Skew-Hermitian = Skew-Hermitian
- Skew-Hermitian $\otimes$ Skew-Hermitian = Hermitian
- Upper-triangular $\otimes$ Upper-triangular = Upper-triangular
- $\frac{d}{dt}(\mathbf{A}(t) \otimes \mathbf{B}(t)) = \frac{d}{dt}\mathbf{A}(t) \otimes \mathbf{B}(t) + \mathbf{A}(t) \otimes \frac{d}{dt}\mathbf{B}(t)$

Exercise 323 Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^H$ and $\mathbf{B} = \mathbf{X}\Lambda\mathbf{Y}^H$ be SVDs. Show that the SVD of $\mathbf{A} \otimes \mathbf{B}$ is given by

$$(\mathbf{U} \otimes \mathbf{X})(\Sigma \otimes \Lambda)(\mathbf{V} \otimes \mathbf{Y})^H.$$

Exercise 324 Show that $\mathbf{rank}(\mathbf{A} \otimes \mathbf{B}) = \mathbf{rank}(\mathbf{A})\,\mathbf{rank}(\mathbf{B})$.

Exercise 325 Let $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$ and $\mathbf{B} = \mathbf{W}\mathbf{G}\mathbf{W}^{-1}$ denote Jordan decompositions. Show that

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{V} \otimes \mathbf{W})(\mathbf{J} \otimes \mathbf{G})(\mathbf{V} \otimes \mathbf{W})^{-1}.$$

Conclude that $\lambda_i(\mathbf{A} \otimes \mathbf{B}) = \lambda_r(\mathbf{A})\lambda_s(\mathbf{B})$. Note that this is not a Jordan decomposition.

Exercise 326 Let $\mathbf{A}$ be an $m \times m$ matrix and $\mathbf{B}$ be an $n \times n$ matrix. Show that

- $\mathbf{trace}(\mathbf{A} \otimes \mathbf{B}) = \mathbf{trace}(\mathbf{A})\,\mathbf{trace}(\mathbf{B})$.
- $(\mathbf{A} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{B}) = \mathbf{A} \otimes \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{I}_n)$.

Exercise 327 Show that

$$\mathbf{diag}\{\mathbf{A}_i\}_{i=1}^n \otimes \mathbf{diag}\{\mathbf{B}_j\}_{j=1}^m = \mathbf{diag}\{\mathbf{diag}\{\mathbf{A}_i \otimes \mathbf{B}_j\}_{j=1}^m\}_{i=1}^n.$$

6.2 Tensor Product Spaces

At this point it is a good idea to look at the vector space structure of tensor products. We will avoid an abstract approach (since I don't want to define dual spaces).

Let $\mathbb{F}^{i_j}$ denote vector spaces for positive integers i_1, i_2, …, i_n. We define the tensor product of these vector spaces via the formula

$$\otimes_{j=1}^{n}\mathbb{F}^{i_j} = \mathbb{F}^{i_1} \otimes \mathbb{F}^{i_2} \otimes \cdots \otimes \mathbb{F}^{i_n} = \mathbf{span}\{\otimes_{j=1}^{n}\mathbf{x}_j \,|\, \mathbf{x}_j \in \mathbb{F}^{i_j},\ j = 1, \ldots, n\}.$$

Remember that **span** only allows finite linear combinations of its elements. Therefore an arbitrary element of $\otimes_j\mathbb{F}^{i_j}$ can be written in the form $\sum_{k=1}^{l} \alpha_k \otimes_{j=1}^{n} \mathbf{x}_{kj}$, where $\mathbf{x}_{kj} \in \mathbb{F}^{i_j}$.

Exercise 328 Show that $\otimes_{j=1}^{n}\mathbb{F}^{i_j}$ is a sub-space of $\mathbb{F}^{\Pi_{j=1}^{n} i_j}$, where $\Pi_{j=1}^{n} i_j = i_1 i_2 \cdots i_n$.

Actually $\otimes_{j=1}^{n}\mathbb{F}^{i_j} = \mathbb{F}^{\Pi_{j=1}^{n} i_j}$. We will prove this by constructing a suitable basis. However, to keep the notation simple we will concentrate on the important case when $i_j = m$ for all j. In this case we will use the notation $\otimes^n\mathbb{F}^m$.

Exercise 329 Show that if $\otimes_i \mathbf{x}_i = \mathbf{0}$ then at least one of $\mathbf{x}_i = \mathbf{0}$.

Exercise 330 Show that there is a vector in $\otimes^2\mathbb{R}^2$ that is *not* of the form $\mathbf{x}\otimes\mathbf{y}$. *Hint*: $\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}^T$.

At this point it is useful to introduce some notation about **multi-indices**. Let $\mathcal{I}$ denote the n-tuple $(i_1, i_2, \ldots, i_n)$ where $1 \le i_j \le m$. We will then use the notation

$$\otimes_{i\in\mathcal{I}}\mathbf{x}_i = \otimes_{j=1}^{n}\mathbf{x}_{i_j}.$$

We will assume that n-tuples $\mathcal{I}$ are ordered lexicographically; that is,

$$(i_1, i_2, \ldots, i_n) < (j_1, j_2, \ldots, j_n),$$

iff $i_k = j_k$ for $k = 1, \ldots, l$, and $i_{l+1} < j_{l+1}$.

Let $\mathbf{e}_i$ denote column i of the identity matrix. The length of $\mathbf{e}_i$ will be apparent from the context. Note that multiple occurences of $\mathbf{e}_i$ in the *same* formula can denote column vcetors of *different* lengths.

It is easy to check that the m^n vectors

$$\otimes_{i\in\mathcal{I}}\mathbf{e}_i = \mathbf{e}_{\mathcal{I}},$$

form an orthonormal basis for $\otimes^n\mathbb{F}^m$.

Exercise 331 Check this claim.

Exercise 332 Write down a basis for $\otimes_{j=1}^{n}\mathbb{F}^{i_j}$ from bases for $\mathbb{F}^{i_j}$.

We are now ready to compute the Jordan decomposition of the tensor product of two nilpotent matrices.

Exercise 333 Show that the smallest integer k for which $(\mathbf{Z}_p \otimes \mathbf{Z}_q)^k = \mathbf{0}$, is $k = \min(p, q)$. *Hint*: $(\mathbf{Z}_p \otimes \mathbf{Z}_q)^r = \mathbf{Z}_p^r \otimes \mathbf{Z}_q^r$.

From now on without loss of generality we will assume $p \leq q$.

Exercise 334 Show that if $\mathbf{v} \in \mathcal{N}(\mathbf{A})$ then $\mathbf{v} \otimes \mathbf{w} \in \mathcal{N}(\mathbf{A} \otimes \mathbf{B})$.

Exercise 335 Show that $\mathbf{Z}_p^{r-1}\mathbf{e}_r \neq \mathbf{0}$, while $\mathbf{Z}_p^r\mathbf{e}_r = \mathbf{0}$. *Hint*: $\mathbf{Z}_p\mathbf{e}_i = \mathbf{e}_{i-1}$.

Therefore $\{\mathbf{Z}_p^k\mathbf{e}_p\}_{k=0}^{p-1}$ forms a right Jordan chain of length p for $\mathbf{Z}_p$.

Exercise 336 Show that $\{(\mathbf{Z}_p \otimes \mathbf{Z}_q)^k(\mathbf{e}_p \otimes \mathbf{e}_r)\}_{k=0}^{p-1}$ forms a right Jordan chain of length p for $p \leq r \leq q$.

This gives us $q - p + 1$ linearly independent right Jordan chains. So there are at least $q - p + 1$ Jordan blocks of size p in the Jordan decomposition of $\mathbf{Z}_p \otimes \mathbf{Z}_q$ when $p \leq q$. In fact there are exactly $q - p + 1$ Jordan blocks of size p. This will become apparent soon. Define the following subspace

$$\mathcal{U}_p = \mathbf{span}\{\mathbf{e}_{p-i} \otimes \mathbf{e}_{r-i} \,|\, i = 0, \ldots, p-1,\ r = p, \ldots, q\}.$$

Note that $\mathbf{dim}(\mathcal{U}_p) = p(q - p + 1)$ and $\mathbf{dim}(\mathcal{U}_p^\perp) = p(p-1)$.

Now consider the two chains $\{(\mathbf{Z}_p \otimes \mathbf{Z}_q)^k(\mathbf{e}_{p-1} \otimes \mathbf{e}_q)\}_{k=0}^{p-2}$ and $\{(\mathbf{Z}_p \otimes \mathbf{Z}_q)^k(\mathbf{e}_p \otimes \mathbf{e}_{p-1})\}_{k=0}^{p-2}$, of length $p-1$. Observe that the starting point of the chains, $\mathbf{e}_{p-1} \otimes \mathbf{e}_q$ and $\mathbf{e}_p \otimes \mathbf{e}_{p-1}$, are not in the subspace $\mathcal{U}_p$, nor are any subsequent members of the chain in $\mathcal{U}_p$. Therefore these are two new chains of length $p-1$ which establishes that there are at least two Jordan blocks of size $p-1$. In fact there are exactly two Jordan blocks of size $p-1$ as will be apparent soon. Define the subspace

$$\begin{aligned}\mathcal{U}_{p-1} = &\,\mathbf{span}\{\mathbf{e}_{p-1-i} \otimes \mathbf{e}_{q-i} \,|\, i = 0, \ldots, p-2\} + \\ &\,\mathbf{span}\{\mathbf{e}_{p-i} \otimes \mathbf{e}_{p-1-i} \,|\, i = 0, \ldots, p-2\}.\end{aligned}$$

Observe that $\mathbf{dim}(\mathcal{U}_{p-1}) = 2(p-1)$ and that $\mathcal{U}_p \perp \mathcal{U}_{p-1}$.

We can continue in this way to define new linearly independent right Jordan chains. In general for any integer $1 \leq r < p$ we define two right Jordan chains, $\{(\mathbf{Z}_p \otimes$

$\mathbf{Z}_q)^k(\mathbf{e}_{p-r} \otimes \mathbf{e}_q)\}_{k=0}^{p-r-1}$ and $\{(\mathbf{Z}_p \otimes \mathbf{Z}_q)^k(\mathbf{e}_p \otimes \mathbf{e}_{p-r})\}_{k=0}^{p-r-1}$, of length $p-r$. Define the subspace

$$\begin{aligned}\mathcal{U}_{p-r} = &\,\mathbf{span}\{\mathbf{e}_{p-r-i} \otimes \mathbf{e}_{q-i} \,|\, i = 0, \ldots, p-r-1\} + \\ &\,\mathbf{span}\{\mathbf{e}_{p-i} \otimes \mathbf{e}_{p-r-i} \,|\, i = 0, \ldots, p-r-1\}.\end{aligned}$$

Observe that $\mathbf{dim}(\mathcal{U}_{p-r}) = 2(p-r)$ and that $\mathcal{U}_s \perp \mathcal{U}_{p-r}$ for $s > p-r$. Therefore there are at least two Jordan blocks of size $p-r$. In fact there are exactly two Jordan blocks of size $p-r$ as will be apparent soon.

Finally observe that

$$\mathbf{dim}(\mathcal{U}_p) + \sum_{r=1}^{p-1} \mathbf{dim}(\mathcal{U}_{p-r}) = p(q-p+1) + \sum_{r=1}^{p-1} 2(p-r) = pq = \mathbf{dim}(\mathbb{C}^p \otimes \mathbb{C}^q).$$

Therefore it follows that we have found a complete set of Jordan chains and all our claims are proved: there are $q-p+1$ Jordan blocks of size p and two Jordan blocks of size 1 through $p-1$.

If you have a nice way to express the Jordan decomposition of $\mathbf{Z}_{p_1} \otimes \cdots \otimes \mathbf{Z}_{p_n}$, and $(\lambda \mathbf{I} + \mathbf{Z}_p) \otimes (\mu \mathbf{I} + \mathbf{Z}_q)$, please let me know.

6.3 Symmetric tensors

The full tensor product spaces are not very interesting since they are the same as (*isomorphic* to) $\mathbb{C}^n$. However, they contain interesting subspaces that occur frequently. We have met some of them already; namely, the class of Hermitian and skew-Hermitian matrices.

Let $\mathcal{P}_n$ denote the set of all permutations of the integers $1, \ldots, n$. Let $\mathbf{x}_i \in \mathbb{R}^m$ for $i = 1, \ldots, n$. We define the symmetric tensor product of $\mathbf{x}_i$ to be

$$\mathbf{x}_1 \vee \mathbf{x}_2 \vee \cdots \vee \mathbf{x}_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{x}_{\sigma(i)}.$$

We denote the sub-space of $\otimes^n \mathbb{R}^m$ spanned by all symmetric tensor products of n vectors from $\mathbb{R}^m$ as $\vee^n \mathbb{R}^m$. We will use the convenient notation $\vee_{i=1}^n \mathbf{x}_i$ for the symmetric tensor product of $\mathbf{x}_i$.

Exercise 337 Show that

$$\mathbf{x}_1 \vee \cdots \vee \mathbf{x}_i \vee \cdots \vee \mathbf{x}_j \vee \cdots \vee \mathbf{x}_n = \mathbf{x}_1 \vee \cdots \vee \mathbf{x}_j \vee \cdots \vee \mathbf{x}_i \vee \cdots \vee \mathbf{x}_n.$$

We will write this fact succinctly as $\vee_{i=1}^n \mathbf{x}_i = \vee_{i=1}^n \mathbf{x}_{\sigma(i)}$ for any permutation $\sigma \in \mathcal{P}_n$. (Prove it.)

Exercise 338 Give an example of $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$, where

$$(\mathbf{x} \vee \mathbf{y}) \otimes \mathbf{z} + \mathbf{z} \otimes (\mathbf{x} \vee \mathbf{y}) \neq c(\mathbf{x} \vee \mathbf{y} \vee \mathbf{z}),$$

for any choice of the constant c. This exercise shows that a naive definition of symmetric tensor product is not associative.

Let $G_{m,n} = \{(i_1, i_2, \ldots, i_n) : 1 \leq i_k \leq i_{k+1} \leq m\}$. That is $G_{m,n}$ is the set of n-tuples with components from the set $\{1, \ldots, m\}$ in non-decreasing order. Remember that we use the notation $\mathcal{I} = (i_1, \ldots, i_n)$ to denote n-tuples. Suppose that there are n_i occurences of the number i in the tuple $\mathcal{I}$. Then we will use the notation $\mathcal{I}! = n_1! n_2! \cdots n_m!$.

We claim that the set of symmetric tensors

$$\mathbf{g}_{\mathcal{I}} = \sqrt{\frac{n!}{\mathcal{I}!}} \vee_i \mathbf{e}_{\mathcal{I}_I}, \quad \mathcal{I} \in G_{m,n},$$

forms an orthonormal basis for $\vee^n \mathbb{R}^m$.

Exercise 339 Show that if $\mathcal{I}, \mathcal{J} \in G_{m,n}$ and $\mathcal{I} \neq \mathcal{J}$ then $\mathbf{g}_{\mathcal{I}}^T \mathbf{g}_{\mathcal{J}} = 0$. *Hint*: Do a small example first.

Next we check that they have unit length. Let $\mathcal{I} = (i_1, \ldots, i_n) \in G_{m,n}$. Then

$$\mathbf{g}_{\mathcal{I}} = \frac{1}{\sqrt{\mathcal{I}!n!}} \sum_{\sigma \in \mathcal{P}_n} \otimes_{k=1}^{n} \mathbf{e}_{\mathcal{I}_{\sigma(k)}}.$$

Therefore

$$\mathbf{g}_{\mathcal{I}}^T \mathbf{g}_{\mathcal{I}} = \frac{1}{\mathcal{I}!n!} \sum_{\sigma,\tau \in \mathcal{P}_n} \otimes_{k=1}^{n} \mathbf{e}_{\mathcal{I}_{\sigma(k)}}^T \mathbf{e}_{\mathcal{I}_{\tau(k)}} = 1.$$

To see this consider a term in the sum for a fixed σ. Clearly the term evaluates to 1 if $\tau = \sigma$. But any τ which only permutes components in $\mathcal{I}$ that are identical among themselves will still yield a term that evaluates to 1. For each σ there are $\mathcal{I}!$ such τ terms. Therefore the right-hand side adds up to 1. This establishes that the $\mathbf{g}_{\mathcal{I}}$ for $\mathcal{I} \in G_{m,n}$ form an orthonormal set.

To finish establishing that it is a basis we must show that they span $\vee^n \mathbb{R}^m$.

Exercise 340 Establish that it is sufficient to show that an *elementary* symmetric tensor, $\vee_{i=1}^{n} \mathbf{x}_i$, can be written as a linear combination of the $\mathbf{g}_{\mathcal{I}}$'s.

Let $F_{m,n}$ denote the set of all n-tuples formed from the integers between 1 and m (inclusive). Then observe that

$$\begin{aligned}
\vee_{i=1}^{n} \mathbf{x}_i &= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \otimes_{l=1}^{n} \mathbf{x}_{\sigma(l)} \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \otimes_{l=1}^{n} \sum_{j=1}^{m} \mathbf{e}_j x_{j,\sigma(l)} \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \sum_{\mathcal{I} \in F_{m,n}} \otimes_{l=1}^{n} \mathbf{e}_{\mathcal{I}_l} x_{\mathcal{I}_l,\sigma(l)} \qquad \text{(why?)} \\
&= \sum_{\mathcal{I} \in F_{m,n}} \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \otimes_{l=1}^{n} \mathbf{e}_{\mathcal{I}_l} x_{\mathcal{I}_l,\sigma(l)} \\
&= \sum_{\mathcal{I} \in F_{m,n}} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_l,\sigma(l)} \right) \otimes_{l=1}^{n} \mathbf{e}_{\mathcal{I}_l}.
\end{aligned}$$

Now observe that for a fixed $\mathcal{I} \in G_{m,n}$ and any $\tau \in \mathcal{P}_n$

$$\sum_{\sigma \in \mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_l,\sigma(l)} = \sum_{\sigma \in \mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_{\tau(l)},\sigma(l)}.$$

However for each $\mathcal{I} \in G_{m,n}$ there are only $n!/\mathcal{I}!$ occurences of $\tau(\mathcal{I})$ in the actual sum. Therefore we can group the terms further together and obtain

$$
\begin{aligned}
\vee_{i=1}^{n}\mathbf{x}_i &= \sum_{\mathcal{I}\in F_{m,n}} \frac{1}{n!}\left(\sum_{\sigma\in\mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_l,\sigma(l)}\right) \otimes_{l=1}^{n} \mathbf{e}_{\mathcal{I}_l} \\
&= \sum_{\mathcal{I}\in G_{m,n}} \frac{1}{n!}\left(\sum_{\sigma\in\mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_l,\sigma(l)}\right) \frac{1}{\mathcal{I}!}\sum_{\tau\in\mathcal{P}_n} \otimes_{l=1}^{n}\mathbf{e}_{\mathcal{I}_{\tau(l)}} \qquad (6.1)\\
&= \sum_{\mathcal{I}\in G_{m,n}} \left(\frac{1}{\mathcal{I}!}\sum_{\sigma\in\mathcal{P}_n} \Pi_{l=1}^{n} x_{\mathcal{I}_l,\sigma(l)}\right) \vee_{l=1}^{n} \mathbf{e}_{\mathcal{I}_l}.
\end{aligned}
$$

Hence we have shown that $\mathbf{g}_{\mathcal{I}}$ for $\mathcal{I}\in G_{m,n}$ is an orthonormal basis for $\vee^n\mathbb{R}^m$.

Therefore $\mathbf{dim}(\vee^n\mathbb{R}^m)$ is the cardinality of the set $G_{m.n}$. Let $s(m,n)$ denote the latter number. Observe that $s(1,n)=1$ and $s(m,1)=m$. Now let us see how we can generate the tuples in $G_{m.n}$ using tuples in $G_{m-1,n}$ and $G_{m,n-1}$. Partition the tuples in $G_{m,n}$ into two sets; let the first set of tuples start with the number 1, and the second set be everything else. Clearly by prepending a 1 to every tuple in $G_{m,n-1}$ we can obtain exactly the first set. Similarly we can obtain the second set by taking every tuple in $G_{m-1,n}$ and adding 1 to every component. Therefore it follows that $s(m,n)=s(m,n-1)+s(m-1,n)$. With the initial conditions $s(1,n)=1$ and $s(m,1)=m$, this recursion uniquely specifies $s(m,n)$ for all positive integers.

Exercise 341 Verify that

$$
\mathbf{dim}(\vee^n\mathbb{R}^m) = s(m,n) = \binom{m+n-1}{n}.
$$

Next we compute the orthogonal projector $P_\vee$, from $\otimes^n\mathbb{R}^m$ onto $\vee^n\mathbb{R}^m$ via its action on the orthogonal basis $\mathbf{e}_{\mathcal{I}}$ for $\mathcal{I}\in F_{m,n}$

$$
P_\vee(\otimes_{i\in\mathcal{I}}\mathbf{e}_i) = \vee_{i\in\mathcal{I}}\mathbf{e}_{i\in\mathcal{I}}.
$$

We begin by checking if $P_\vee$ is idempotent. Clearly it is sufficient to check if $P_\vee\mathbf{g}_{\mathcal{I}} = \mathbf{g}_{\mathcal{I}}$ for $\mathcal{I}\in G_{m,n}$. Observe that

$$
\begin{aligned}
P_\vee\left(\frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n}\otimes_{i=1}^{n}\mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right) &= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} P_\vee\left(\otimes_{i=1}^{n}\mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right)\\
&= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \vee_{i=1}^{n}\mathbf{e}_{\mathcal{I}_{\sigma(i)}}\\
&= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \vee_{i=1}^{n}\mathbf{e}_{\mathcal{I}_i}\\
&= \vee_{i=1}^{n}\mathbf{e}_{\mathcal{I}_i},
\end{aligned}
$$

which proves that $P_\vee$ is idempotent. This also explains the presence of the factor $n!$ in the definition of the symmetric tensor product $\vee$.

Finally we check if $\mathbf{x} - P_\vee \mathbf{x}$ is perpendicular to $P_\vee \mathbf{x}$ for all $\mathbf{x} \in \otimes^n \mathbb{R}^m$. It is sufficient to check that $\otimes_i \mathbf{e}_{\mathcal{I}_i} - P_\vee(\otimes_i \mathbf{e}_{\mathcal{I}_i})$ is perpendicular to $\mathbf{g}_{\mathcal{J}}$ for $\mathcal{I} \in F_{m,n}$ and $\mathcal{J} \in G_{m,n}$. We break the calculation up into 2 cases. First we assume that there is no permutation σ such that $\sigma(\mathcal{I}) = \mathcal{J}$. Then clearly

$$\left(\frac{1}{n!}\sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{e}_{\mathcal{J}_{\sigma(i)}}\right)^T \left(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} - \frac{1}{n!}\sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right) = 0.$$

Next we consider the case when $\tau(\mathcal{I}) = \mathcal{J}$ for some $\tau \in \mathcal{P}_n$. Then we have that

$$\left(\frac{1}{n!}\sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{e}_{\mathcal{J}_{\sigma(i)}}\right)^T \left(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} - \frac{1}{n!}\sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right) = \frac{\mathcal{J}!}{n!} - \frac{1}{(n!)^2}\mathcal{J}!n!.$$

Therefore we have shown that $P_\vee$ is an orthogonal projector onto $\vee^n \mathbb{R}^m$.

For $\mathcal{I} \in G_{m,n_1}$ and $\mathcal{J} \in G_{m,n_2}$ we have by an easy calculation that

$$P_\vee\left(\left(\frac{1}{n_1!}\sum_{\sigma \in \mathcal{P}_{n_1}} \otimes_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right) \otimes \left(\frac{1}{n_2!}\sum_{\tau \in \mathcal{P}_{n_2}} \otimes_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_{\tau(i)}}\right)\right) = \vee_{i=1}^{n_1+n_2} \mathbf{e}_{(\mathcal{I},\mathcal{J})_i}.$$

Hence we can extend the definition of $\vee$, the symmetric tensor product, to a binary operator between two symmetric tensors by first defining it on bases for $\vee^n \mathbb{R}^m$:

$$\left(\vee_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}\right) \vee \left(\vee_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_i}\right) = P_\vee\left(\left(\vee_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}\right) \otimes \left(\vee_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_i}\right)\right) = \vee_{i=1}^{n_1+n_2} \mathbf{e}_{(\mathcal{I},\mathcal{J})_i}.$$

More generally for $\mathbf{x} \in \vee^{n_1} \mathbb{R}^m$ and $\mathbf{y} \in \vee^{n_2} \mathbb{R}^m$, we have

$$\mathbf{x} = \sum_{\mathcal{I} \in G_{m,n_1}} x_{\mathcal{I}} \vee_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}, \qquad \text{and} \qquad \mathbf{y} = \sum_{\mathcal{I} \in G_{m,n_2}} y_{\mathcal{I}} \vee_{i=1}^{n_2} \mathbf{e}_{\mathcal{I}_i}.$$

Hence

$$\mathbf{x} \vee \mathbf{y} = P_\vee(\mathbf{x} \otimes \mathbf{y}) = \sum_{\substack{\mathcal{I} \in G_{m,n_1} \\ \mathcal{J} \in G_{m,n_2}}} x_{\mathcal{I}}\, y_{\mathcal{J}} \vee_{i=1}^{n_1+n_2} \mathbf{e}_{(\mathcal{I},\mathcal{J})_i}.$$

Exercise 342 Show that for symmetric tensors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and scalar α

- $(\mathbf{x} + \alpha\mathbf{z}) \vee \mathbf{y} = \mathbf{x} \vee \mathbf{y} + \alpha(\mathbf{z} \vee \mathbf{y})$
- $\mathbf{x} \vee \mathbf{y} = \mathbf{y} \vee \mathbf{x}$
- $(\mathbf{x} \vee \mathbf{y}) \vee \mathbf{z} = \mathbf{x} \vee (\mathbf{y} \vee \mathbf{z})$

Exercise 343 Let $\mathbf{x}_i = \mathbf{x}$ for $i = 1, \ldots, n$. Show that $\otimes_{i=1}^n \mathbf{x}_i = \otimes^n \mathbf{x} = \vee_{i=1}^n \mathbf{x}_i = \vee^n \mathbf{x}$.

An instant question is whether $\mathbf{span}\{\otimes^n \mathbf{x} : \mathbf{x} \in \mathbb{R}^m\} = \vee^n \mathbb{R}^m$?. The answer is yes. To see this note that it is sufficient to show that an arbitrary basis element $\vee_i \mathbf{e}_{\mathcal{I}_i}$ for some $\mathcal{I} \in G_{m,n}$ is in the span. Without loss of generality assume that $\mathcal{I}$ only contains the first k integers from 1 to k. In particular let us assume that the number i occurs exactly j_i times in $\mathcal{I}$. We will show that this basis vector can be written as a linear combination of the symmetric tensors $\vee^n(\sum_{i=1}^k \alpha_i \mathbf{e}_i)$ for suitable choice of α_i. To make this calculation easier we will exploit the fact that the symmetric tensor product between symmetric tensors is commutative, associative and distributive and write $\mathbf{x} \vee \mathbf{y}$ as $\mathbf{xy}$ whenever $\mathbf{x}$ and $\mathbf{y}$ are symmetric tensors. Therefore we have that $\vee^n \mathbf{x} = \mathbf{x}^n$, for example. Observe that

$$\left(\sum_{i=1}^k \alpha_i \mathbf{e}_i\right)^n = \sum_{i_1+i_2+\cdots+i_k=0}^{n} \frac{n!}{i_1! i_2! \cdots i_k!} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_k^{i_k} \mathbf{e}_1^{i_1} \mathbf{e}_2^{i_2} \cdots \mathbf{e}_k^{i_k}.$$

Now we take a linear combination of $N = (n+1)^k$ of these terms and obtain

$$\sum_{p=1}^N \beta_p \left(\sum_{i=1}^k \alpha_{p,i} \mathbf{e}_i\right)^n = \sum_{i_1+i_2+\cdots+i_k=0}^{n} \frac{n!}{i_1! i_2! \cdots i_k!} \mathbf{e}_1^{i_1} \mathbf{e}_2^{i_2} \cdots \mathbf{e}_k^{i_k} \sum_{p=1}^N \beta_p \alpha_{p,1}^{i_1} \alpha_{p,2}^{i_2} \cdots \alpha_{p,k}^{i_k}.$$

Therefore to recover just the term with $i_l = j_l$ we must pick β_p and $\alpha_{i,p}$ such that

$$\sum_{p=1}^N \beta_p \alpha_{p,1}^{i_1} \alpha_{p,2}^{i_2} \cdots \alpha_{p,k}^{i_k} = \begin{cases} 0, & \text{if } (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k), \\ 1, & \text{if } (i_1, \ldots, i_k) = (j_1, \ldots, j_k). \end{cases}$$

We pick $\alpha_{p,1} = 1$ and $\alpha_{p,i} = x_p$, where

$$x_0 < x_1 < \cdots < x_N.$$

We then observe that β_p is obtained by solving an adjoint multi-dimensional Vandermonde system, which, with our choice of $\alpha_{p,i}$ is known to be invertible. In particular the coefficient matrix can be written as k-th tensor power of a $(n+1) \times (n+1)$ Vandermonde matrix. This establishes our claim.

Inner products of elementary symmetric tensors are given by the permanents of certain matrices.

Permanent The permanent of an $n \times n$ matrix is defined to be **109**

$$\mathbf{per}(\mathbf{A}) = \sum_{\sigma \in \mathcal{P}_n} \Pi_{i=1}^n \mathbf{A}_{i,\sigma(i)}.$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be $m \times n$ matrices. We will use the notation $\mathbf{X}_i$ to denote column i of $\mathbf{X}$. We now show that

$$(\vee_{i=1}^n \mathbf{X}_i)^T (\vee_{i=1}^n \mathbf{Y}_i) = \frac{1}{n!} \mathbf{per}(\mathbf{X}^T \mathbf{Y}).$$

We calculate as follows

$$\begin{aligned}(\vee_{i=1}^n \mathbf{X}_i)^T (\vee_{i=1}^n \mathbf{Y}_i) &= \frac{1}{(n!)^2} \left(\sum_{\sigma \in \mathcal{P}_n} \otimes_i \mathbf{X}_{\sigma(i)}^T \right) \left(\sum_{\tau \in \mathcal{P}_n} \otimes_i \mathbf{Y}_{\tau(i)} \right) \\ &= \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{P}_n} \Pi_{i=1}^n \mathbf{X}_i^T \mathbf{Y}_{\sigma(i)} \right),\end{aligned}$$

which proves the claim.

Exercise 344 Observe that in equation 6.1 we give an explicit formula to expand a symmetric tensor in terms of $\vee_{i \in \mathcal{I}} \mathbf{e}_i$ for $\mathcal{I} \in G_{m,n}$. The above formula can also be used for this purpose by choosing for example $\mathbf{Y}_i = \mathbf{e}_{\mathcal{I}_i}$. However there seems to be an extra $\mathcal{I}!$ in one of the formulas. Can you reconcile them?

Exercise 345 Show that $\mathbf{per}(\mathbf{X}^T \mathbf{X}) \geq 0$.

Exercise 346 Show that

$$|\,\mathbf{per}(\mathbf{X}^T \mathbf{Y})| \leq \sqrt{\mathbf{per}(\mathbf{X}^T \mathbf{X})\, \mathbf{per}(\mathbf{Y}^T \mathbf{Y})}.$$

By placing restrictions on the basis set we can get lower dimensional symmetric subspaces. Let $\mathbf{U} = (\,\mathbf{U}_1 \quad \mathbf{U}_2\,)$, be an orthogonal $m \times m$ matrix with $\mathbf{U}_1$ containing m_1 columns. Let $\mathbf{u}_i$ denote the columns of $\mathbf{U}$. Denote

$$\mathbf{span}\{\vee_{i=1}^n \mathbf{u}_{\mathcal{I}_i} | \mathcal{I} \in G_{m_1,n}\} = \vee^n \mathcal{R}(\mathbf{U}_1).$$

Note that $\vee^n \mathcal{R}(\mathbf{U}_1)$ is a subspace of $\vee^n \mathbb{R}^m$.

Exercise 347 Show that $\mathbf{dim}(\vee^n \mathbb{R}^{m_1}) = \mathbf{dim}(\vee^n \mathcal{R}(\mathbf{U}_1))$.

Denote

$$\mathbf{span}\{\mathbf{x} \vee \mathbf{y} | \mathbf{x} \in \vee^{n_1} \mathcal{R}(\mathbf{U}_1), \mathbf{y} \in \vee^{n_2} \mathcal{R}(\mathbf{U}_2)\} = (\vee^{n_1} \mathcal{R}(\mathbf{U}_1)) \vee (\vee^{n_2} \mathcal{R}(\mathbf{U}_2)).$$

Exercise 348 Show that $\mathbf{dim}((\vee^{n_1} \mathcal{R}(\mathbf{U}_1)) \vee (\vee^{n_2} \mathcal{R}(\mathbf{U}_2))) = \mathbf{dim}(\vee^{n_1} \mathcal{R}(\mathbf{U}_1))\, \mathbf{dim}(\vee^{n_2} \mathcal{R}(\mathbf{U}_2))$.

Exercise 349 Show that

$$\vee^n \mathbb{R}^m = \oplus_{j=0}^n (\vee^j \mathcal{R}(\mathbf{U}_1)) \vee (\vee^{n-j} \mathcal{R}(\mathbf{U}_2)).$$

Cross check by verifying independently that

$$\binom{m_1+m_2+n-1}{n} = \sum_{j=0}^{n} \binom{m_1+j-1}{j} \binom{m_2+n-j-1}{n-j}.$$

Hint: To proceed first extend the sum to

$$\binom{m_1+m_2+n-1}{n} = \sum_{j=0}^{m_2+n-1} \binom{m_1+j-1}{j} \binom{m_2+n-j-1}{n-j},$$

and then convert it to

$$\binom{m_1+m_2+n-1}{n} = \sum_{j=0}^{m_2+n-1} \binom{m_1+j-1}{m_1-1} \binom{m_2+n-j-1}{m_2-1}.$$

Now use identity (5.26) from *Concrete Mathematics* by Graham, Knuth and Patashnik.

Finally all of these formulas remain true if we merely require that $\mathbf{U}$ is non-singular. Verify.

It is also convenient to be detect a symmetric tensor from its coefficients in the canonical basis $\mathbf{e}_\mathcal{I}$ for $\mathcal{I} \in F_{m,n}$. Let $\mathbf{x} = \sum_{\mathcal{I}\in F_{m,n}} \tilde{x}_\mathcal{I} \mathbf{e}_\mathcal{I} = \sum_{\mathcal{J}\in G_{m,n}} x_\mathcal{J} \mathbf{g}_\mathcal{J}$.

Exercise 350 Show that $\tilde{x}_{\eta(\mathcal{I})} = \tilde{x}_\mathcal{I}$ for all exchange permutations η.

Exercise 351 Conclude that $\tilde{x}_{\sigma(\mathcal{I})} = \tilde{x}_\mathcal{I}$ for all permutations σ.

This explains why symmetric tensors form such a small subspace of $\otimes^n \mathbb{R}^m$. This is also an exact characterization of symmetric tensors.

Exercise 352 Show that $\mathbf{x} = \sum_{\mathcal{I}\in F_{m,n}} x_\mathcal{I} \mathbf{e}_\mathcal{I} \in \vee^n \mathbb{R}^m$ iff $x_{\sigma(\mathcal{I})} = x_\mathcal{I}$ for all permutations σ.

Therefore we can characterize the symmetric tensors as those $\mathbf{x} = \sum_{\mathcal{I}\in F_{m,n}} x_\mathcal{I} \mathbf{e}_\mathcal{I}$ that are in the nullspace of the equations

$$x_\mathcal{I} = x_{\eta(\mathcal{I})}, \qquad \text{for all exchanges } \eta \text{ and all } \mathcal{I} \in G_{m,n}.$$

One is then lead to consider other "symmetry" conditions on the tensor. Here is a problem from Bishop and Goldberg.

Example 6 Find all $\mathbf{x} = \sum_{i,j,k=1}^{3} x_{i,j,k}\mathbf{e}_{(i,j,k)} \in \otimes^3\mathbb{R}^m$ that satisfy the "symmetry" equations

$$x_{i,j,k} + x_{i,k,j} = 0$$
$$x_{i,j,k} + x_{j,k,i} + x_{k,i,j} = 0,$$

for $i, j, k = 1$ to m. The first set of equations imply that the free variables can be chosen from the set $x_{i,j,k}$ with $1 \leq j < k \leq m$. Of course $x_{i,j,j} = 0$. This only leaves the second set of equations. We now claim that we can pick only the variables $x_{i,j,k}$ with $1 \leq j < k \leq m$ and $1 \leq i \leq k \leq m$ as free. First let us check if a variable $x_{p,q,r}$ which does not satisfy the conditions, that is $q < r < p$, can be determined from the putative free variables. Observe that

$$x_{p,q,r} = -x_{q,r,p} + x_{r,q,p},$$

and all the variables on the right are free, since $q, r < p$. Obviously a variable $x_{p,q,r}$ with $r < q$ is determined by $x_{p,r,q}$. Further those with $r = q$ are zero. Hence we see that all variables are determined by the free variables. The question is are all equations simulatenously satisfied; that is, did we pick too many free variables. We see that the first set of equations is consistent with our choice as they each determine exactly one basic variable. For the second set, for each choice of triplet (p, q, r) there is an equation

$$x_{p,q,r} + x_{q,r,p} + x_{r,p,q} = 0.$$

If all 3 integers are distinct then there is exactly one basic variable which does not appear in any other such equation. If two of the integers are the same then we repeat a previous anti-symmetry equation. If all three integers are same that variable is 0. So we see the free variables leave all the equations consistently true.

Now we look at a more complicated problem from Bishop and Goldberg. This concerns the symmetry conditions satisfied by the Riemannian curvature tensor.

Example 7 Consider all $\mathbf{x} = \sum_{i,j,k,l=1}^{m} x_{i,j,k,l}\mathbf{e}_{i,j,k,l} \in \otimes^4\mathbb{R}^m$ that satisfy the "symmetry" conditions

1. $x_{i,j,k,l} = -x_{j,i,k,l}$
2. $x_{i,j,k,l} = -x_{i,j,l,k}$
3. $x_{i,j,k,l} + x_{i,k,l,j} + x_{i,l,j,k} = 0$

We first show that any such tensor must automatically satisfy an extra symmetry condition: $x_{i,j,k,l} = x_{k,l,i,j}$. To see this first observe that

$$\begin{aligned} x_{i,j,k,l} &= -x_{i,k,l,j} - x_{i,l,j,k} \\ &= x_{k,i,l,j} + x_{l,i,j,k} \\ &= -x_{k,l,j,i} - x_{k,j,i,l} - x_{l,j,k,i} - x_{l,k,i,j} \\ &= 2x_{k,l,i,j} + x_{k,j,l,i} + x_{l,j,i,k}. \end{aligned}$$

Next we do a similar derivation with a slight modification

$$\begin{aligned} x_{i,j,k,l} &= -x_{j,i,k,l} \\ &= 2x_{k,l,i,j} + x_{k,i,j,l} + x_{l,i,k,j}. \end{aligned}$$

Adding up these two formulae we get

$$\begin{aligned} 2x_{i,j,k,l} &= 4x_{k,l,i,j} + x_{k,j,l,i} + x_{k,i,j,l} + x_{l,j,i,k} + x_{l,i,k,j} \\ &= 4x_{k,l,i,j} - x_{k,l,i,j} - x_{l,k,j,i}, \end{aligned}$$

which proves our claim. Next we establish that if $\mathbf{x}^T(\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{v}\otimes\mathbf{w}) = 0$ for all choices of $\mathbf{v}$ and $\mathbf{w}$ then $\mathbf{x} = \mathbf{0}$. First observe that if $\mathbf{v} = \sum_{i=1}^m v_i\mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^m w_i\mathbf{e}_i$ then

$$\mathbf{x}^T(\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{v}\otimes\mathbf{w}) = \sum_{i,j,k,l=1}^{m} x_{i,j,k,l}v_iw_jv_kw_l = 0.$$

We already know from the skew-symmetry conditions on the first two and last two variables that $x_{iikl} = x_{ijkk} = 0$. Now fix (i,j,k,l) and choose $\mathbf{v} = \mathbf{e}_i$ and $\mathbf{w} = \mathbf{e}_k$. Then the above equation becomes

$$x_{ikik} = 0.$$

Next choose $\mathbf{v} = \mathbf{e}_i$ and $\mathbf{w} = \mathbf{e}_k + \mathbf{e}_l$. Then using the above symmetry condition we have that

$$\begin{aligned} x_{ikik} + x_{ikil} + x_{ilil} + x_{ilik} &= 0 \\ x_{ikil} + x_{ilik} &= 0. \end{aligned}$$

But we have also established that $x_{ikil} - x_{ilik} = 0$. Therefore we can conclude that $x_{ikil} = 0$. By a similar reasoning we can also establish that $x_{ikjk} = 0$. Therefore we have now shown that variables with two or more identical indices in any position will be 0. So the only non-zero variables are those that have four distinct integers for their indices. Therefore consider $\mathbf{v} = \mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{w} = \mathbf{e}_k + \mathbf{e}_l$. Then we have that

$$\begin{aligned} x_{ikjl} + x_{jkil} + x_{iljk} + x_{jlik} &= 0 \\ -x_{ijlk} - x_{ilkj} - x_{jilk} - x_{jlki} + x_{iljk} + x_{jlik} &= 0 \\ x_{iljk} + x_{jlik} &= 0. \end{aligned}$$

This shows that we have skew-symmetry for the second and third variables also, and an application of the skew-symmetry for the first two and last two indices, shows that we have skew-symmetry between the first and fourth indices also. In summary we have shown skew-symmetry between any two pairs of indices. Now we go back to the original symmetry condition and exploit this additional skew-symmetry.

$$\begin{aligned} x_{ijkl} + x_{iklj} + x_{iljk} &= 0 \\ x_{ijkl} + x_{ijkl} + x_{ijkl} &= 0, \end{aligned}$$

which proves our claim. This shows that the tensor satisfying such symmetry conditions must be a subspace of the subspace spanned by all tensors of the form $\mathbf{v} \otimes \mathbf{w} \otimes \mathbf{v} \otimes \mathbf{w}$. The containment is strict since such tensors do not have a skew-symmetry between the first two and last two indices. Finally we show that such tensors can be constructed out of symmetric matrices. Let $b_{ij} = b_{ji}$. We claim that

$$x_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk},$$

is a tensor with the symmetries of a Riemann curvature tensor. The requisite symmetry conditions are easily verified to be true.

A good example of use of symmetric tensors is a Taylor series expansion of a function of several variables. Let $f : \mathbb{R}^m \to \mathbb{R}$ be an analytic real-valued function of m real variables. Define the n-th derivative of f to be a symmetric tensor of order n via

$$\partial^n f(x_1, \ldots, x_m) = \sum_{\mathcal{I} \in G_{m,n}} \frac{\partial^n f}{\partial x_{\mathcal{I}_1} \partial x_{\mathcal{I}_2} \cdots \partial x_{\mathcal{I}_n}} \vee_{i \in \mathcal{I}} \mathbf{e}_i.$$

Exercise 353 Write out $\partial^2 f$ explicitly. Note that it differs from the Hessian of f by a factor of 2.

The reason for representing the partial derivatives as a symmetric tensor should be obvious now. For example, if f is sufficiently nice then

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1},$$

and this is the reason why $\partial^2 f$ is represented as a symmetric tensor.

By considering the Taylor series expansion in t of $f(\mathbf{a} + t\mathbf{x})$ it can be shown that

$$f(\mathbf{a} + \mathbf{x}) = f(\mathbf{a}) + \sum_{n=1}^{\infty} \frac{(\partial^n f(\mathbf{a}))^T}{n!} \otimes^n \mathbf{x}.$$

Exercise 354 Show it assuming that f is sufficiently nice.

An interesting exercise is to compute the Taylor series expansion under an affine linear change of variables. Let $\phi(\mathbf{b}+\mathbf{y}) = \mathbf{a} + \mathbf{A}\mathbf{y}$. Let $g = f \circ \phi$. Clearly

$$g(\mathbf{b}+\mathbf{y}) = g(\mathbf{b}) + \sum_{n=1}^{\infty} \frac{(\partial^n g(\mathbf{b}))^T}{n!} \otimes^n \mathbf{y}.$$

But we would like to express this in terms of f. Observe that

$$g(\mathbf{b}+\mathbf{y}) = f(\mathbf{a}+\mathbf{A}\mathbf{y}) = f(\mathbf{a}) + \sum_{n=1}^{\infty} \frac{(\partial^n f(\mathbf{a}))^T}{n!} \otimes^n \mathbf{A} \otimes^n \mathbf{y},$$

which shows immediately that

$$\partial^n g(\mathbf{b}) = \left(\otimes^n \mathbf{A}^T\right) \partial^n f(\mathbf{a}),$$

whenever $g(\mathbf{b}+\mathbf{y}) = f(\mathbf{a}+\mathbf{A}\mathbf{y})$. A more detailed view of this operation is presented in the next section.

6.4 Symmetric tensor powers

In the last section we saw how tensor powers arose naturally. In this section we look at them more carefully. Let $\mathbf{A}$ denote a $l \times m$ matrix. It is clear that $\otimes^n \mathbf{A}$ can act on $\otimes^n \mathbb{R}^m$ to yield a tensor in $\otimes^n \mathbb{R}^l$ via the usual matrix multiplication

$$(\otimes^n \mathbf{A})(\otimes_{i=1}^n \mathbf{x}_i) = \otimes_{i=1}^n \mathbf{A}\mathbf{x}_i.$$

A simple calculation shows that $\vee^n \mathbb{R}^m$ is an invariant subspace of $\otimes^n \mathbf{A}$ for any $m \times m$ matrix $\mathbf{A}$. It is therefore natural to study the restriction of $\otimes^n \mathbf{A}$ to this subspace. This restricted operator is denoted by $\vee^n \mathbf{A}$ and called the symmetric tensor power of $\mathbf{A}$. More prosaically, let $\mathbf{G}_{m.n}$ denote the matrix whose columns are formed from the orthonormal symmetric tensor basis $\mathbf{g}_{\mathcal{I}}$ for $\mathcal{I} \in G_{m,n}$. Then the invariance of $\vee^n \mathbb{R}^m$ under $\otimes^n \mathbf{A}$ can be written as the equation

$$(\otimes^n \mathbf{A})\mathbf{G}_{m,n} = \mathbf{G}_{m,n}(\vee^n \mathbf{A}).$$

Using the orthonormality of the columns of $\mathbf{G}_{m,n}$ we can infer from this an explicit expression for $\vee^n \mathbf{A}$

$$\vee^n \mathbf{A} = \mathbf{G}_{m,n}^T (\otimes^n \mathbf{A})\mathbf{G}_{m,n}.$$

We will also use the notation

$$\mathbf{G}_{m,n}\mathbf{x}_\vee = \mathbf{x}, \quad \text{for } \mathbf{x} \in \vee^n \mathbb{R}^m.$$

Clearly

$$(\vee^n \mathbf{A})(\vee_{i=1}^n \mathbf{x}_i)_\vee = (\vee_{i=1}^n \mathbf{A}\mathbf{x}_i)_\vee.$$

We start with a simple sequence of calculations

$$\begin{aligned}
(\otimes^n \mathbf{A})(\otimes^n \mathbf{B}) &= \otimes^n (\mathbf{AB}) \\
(\otimes^n \mathbf{A})(\otimes^n \mathbf{B})\mathbf{G}_{m,n} &= (\otimes^n (\mathbf{AB}))\mathbf{G}_{m,n} \\
(\otimes^n \mathbf{A})\mathbf{G}_{m,n}(\vee^n \mathbf{B}) &= \mathbf{G}_{m,n}(\vee^n (\mathbf{AB})) \\
\mathbf{G}_{m,n}(\vee^n \mathbf{A})(\vee^n \mathbf{B}) &= \mathbf{G}_{m,n}(\vee^n (\mathbf{AB})).
\end{aligned}$$

From which, using the full column-rank of $\mathbf{G}_{m,n}$ we can infer that

$$(\vee^n \mathbf{A})(\vee^n \mathbf{B}) = \vee^n (\mathbf{AB}).$$

It is also possible to show that

- $(\vee^n \mathbf{A})^T = \vee^n \mathbf{A}^T$.
- $(\vee^n \mathbf{A})^{-1} = \vee^n \mathbf{A}^{-1}$.

- If $\mathbf{A}$ is either Hermitian, unitary or normal, then so is $\vee^n \mathbf{A}$.
- If $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, for $i = 1, \ldots, n$, with repetitions allowed, then

$$(\vee^n \mathbf{A})(\vee_{i=1}^n \mathbf{v}_i)_\vee = (\Pi_{i=1}^n \lambda_i)(\vee_{i=1}^n \mathbf{v}_i)_\vee.$$

- Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ be the SVD of $\mathbf{A}$. Then

$$\vee^n \mathbf{A} = (\vee^n \mathbf{U})(\vee^n \Sigma)(\vee^n \mathbf{V})^T$$

  is the SVD of $\vee^n \mathbf{A}$.

At this stage it is not clear that $\vee^n \Sigma$ is a diagonal matrix. So we compute an explicit formula for the entries of $\vee^n \mathbf{A}$. Observe that for $\mathcal{I}, \mathcal{J} \in G_{m,n}$

$$\begin{aligned}
(\vee^n \mathbf{A})_{\mathcal{I},\mathcal{J}} &= \mathbf{g}_{\mathcal{I}}^T (\otimes^n \mathbf{A}) \mathbf{g}_{\mathcal{J}} \\
&= \frac{n!}{\sqrt{\mathcal{I}!\mathcal{J}!}} \left(\vee_{i=1}^n \mathbf{e}_{\mathcal{I}_i}^T\right) (\otimes^n \mathbf{A}) \left(\vee_{i=1}^n \mathbf{e}_{\mathcal{J}_i}\right) \\
&= \frac{n!}{\sqrt{\mathcal{I}!\mathcal{J}!}} \left(\vee_{i=1}^n \mathbf{e}_{\mathcal{I}_i}^T\right) \left(\vee_{i=1}^n (\mathbf{A}\mathbf{e}_{\mathcal{J}_i})\right) \\
&= \frac{1}{n!\sqrt{\mathcal{I}!\mathcal{J}!}} \left(\sum_{\sigma \in \mathcal{P}_n} \otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}}^T\right) \left(\sum_{\tau \in \mathcal{P}_n} \otimes_{i=1}^n (\mathbf{A}\mathbf{e}_{\mathcal{J}_{\tau(i)}})\right) \\
&= \frac{1}{\sqrt{\mathcal{I}!\mathcal{J}!}} \left(\sum_{\sigma \in \mathcal{P}_n} \Pi_{i=1}^n \mathbf{A}_{\mathcal{I}_i, \mathcal{J}_{\sigma(i)}}\right).
\end{aligned}$$

Let us define $\mathbf{A}[\mathcal{I}|\mathcal{J}]$ to be the $n \times n$ matrix whose (i, j) element is $\mathbf{A}_{\mathcal{I}_i, \mathcal{J}_j}$. Then we can summarise our formula for $\vee^n \mathbf{A}$ as

$$(\vee^n \mathbf{A})_{\mathcal{I},\mathcal{J}} = \frac{1}{\sqrt{\mathcal{I}!\mathcal{J}!}} \mathbf{per}(\mathbf{A}[\mathcal{I}|\mathcal{J}]).$$

From this formula it is easy to see that the symmetric tensor product of a diagonal matrix is another diagonal matrix and that indeed $\vee^n \Sigma$ contains the singular values of $\vee^n \mathbf{A}$.

6.5 Signs of permutations

Before we proceed we need to discuss the sign of a permutation. Let σ denote a permutation of the integers $1, \ldots, n$. The sign of σ, denoted $\mathbf{sgn}(\sigma)$, is defined to be either $+1$ or -1: it is $+1$ if σ can be represented as the composition of an even number of exchanges; otherwise it is defined to be -1.

Let $\eta_{i,j}$ denote the exchange which switches the position of the i-th and j-th integers. Suppose

$$\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1,$$

is a permutation of $\{1, 2, 3\}$, then we can decompose σ as

$$\sigma = \eta_{1,2} \circ \eta_{1,3},$$

and hence $\mathbf{sgn}(\sigma) = +1$ in this case. The natural question is whether $\mathbf{sgn}$ is well-defined; can a permutation be written as both an odd number of exchanges and an even number of exchanges? No.

A nice proof of this is given in Herstein's *Topics in Algebra*. Let x_i, for $i = 1, \ldots, n$, denote n distinct numbers in increasing order $x_i < x_{i+1}$. For a permutation σ of $\{1, \ldots, n\}$ consider the number

$$\tau(\sigma) = \mathbf{sgn}(\Pi_{i<j}(x_{\sigma(j)} - x_{\sigma(i)})).$$

It is easy to see that τ of the identity permutation is 1. Let $\eta_{i,j}$ denote a permutation that exchanges the number i with the number j. We claim that $\tau(\eta_{i,j} \circ \sigma) = \tau(\sigma \circ \eta_{i,j}) = -\tau(\sigma)$. We compare the terms in the two formulas

$$\tau(\sigma) = \Pi_{r=2}^{n} \Pi_{s=1}^{r-1} (x_{\sigma(r)} - x_{\sigma(s)}),$$
$$\tau(\sigma \circ \eta_{i,j}) = \Pi_{r=2}^{n} \Pi_{s=1}^{r-1} (x_{\sigma(\eta_{i,j}(r))} - x_{\sigma(\eta_{i,j}(s))}).$$

Without loss of generality let $i < j$ and $s < r$. We observe that if neither r nor s is equal to i or j, then

$$x_{\sigma(\eta_{i,j}(r))} - x_{\sigma(\eta_{i,j}(s))} = x_{\sigma(r)} - x_{\sigma(s)}.$$

So any change in sign must be induced by the other terms. First consider the terms where $s_1 < i = r_1$ and $s_2 < i < j = r_2$. We note that these terms can be paired up as follows

$$x_{\sigma(\eta_{i,j}(i))} - x_{\sigma(\eta_{i,j}(s_1))} = x_{\sigma(j)} - x_{\sigma(s_2)}, \qquad s_1 = s_2.$$

Hence they do not induce a net sign change either. Next consider the terms of the form $i = s_1 < r_1 < j$ and $i < s_2 < j = r_2$. These terms can be paired up as follows

$$x_{\sigma(\eta_{i,j}(r_1))} - x_{\sigma(\eta_{i,j}(i))} = x_{\sigma(r_1)} - x_{\sigma(j)} = (-1)(x_{\sigma(j)} - x_{\sigma(s_2)}), \quad s_2 = r_1.$$

Therefore each of these terms cause a sign change. The total sign change is given by $(-1)^{j-i-1}$. Next we consider the terms of the form $i < s_1 < j = r_1$ and $i = s_2 < r_2 < j$. These terms can be paired up as

$$x_{\sigma(\eta_{i,j}(j))} - x_{\sigma(\eta_{i,j}(s_1))} = x_{\sigma(i)} - x_{\sigma(s_1)} = (-1)(x_{\sigma(r_2)} - x_{\sigma(i)}), \quad s_1 = r_2.$$

Therefore these terms cause a total sign change of $(-1)^{j-i-1}$ too. Next we consider the terms of the form $i = s_1 < j < r_1$ and $j = s_2 < r_2$. These can be paired up as

$$x_{\sigma(\eta_{i,j}(r_1))} - x_{\sigma(\eta_{i,j}(i))} = x_{\sigma(r_1)} - x_{\sigma(j)} = x_{\sigma(r_2)} - x_{\sigma(j)}, \quad r_1 = r_2.$$

So these cause no sign change. Next we consider terms of the form $j = s_1 < r_1$ and $i = s_2 < j < r_2$. As in the previous argument there is no sign change for these forms. All of the forms we have considered so far give together no sign change. This leaves us only with the following two terms to compare

$$x_{\sigma(\eta_{i,j}(i))} - x_{\sigma(\eta_{i,j}(j))} = x_{\sigma(j)} - x_{\sigma(i)} = (-1)(x_{\sigma(i)} - x_{\sigma(j)}).$$

Therefore we have exactly one sign change and we have shown that $\tau(\sigma \circ \eta_{i,j}) = -\tau(\sigma)$. The other version $\tau(\sigma \circ \eta_{i,j}) = -\tau(\sigma)$, is proved similarly.

Exercise 355 Do it.

Exercise 356 Show that $\mathbf{sgn}(\sigma)$ is well-defined for permutations σ.

Exercise 357 Show that $\mathbf{sgn}(\sigma) = \mathbf{sgn}(\sigma^{-1})$ for permutations σ.

Exercise 358 Let $\mathcal{I}$ denote an r-tuple and $\mathcal{J}$ an s-tuple and $(\mathcal{I}, \mathcal{J})$ the $r+s$-tuple obtained by concatenating $\mathcal{I}$ and $\mathcal{J}$. Let $\sigma \in \mathcal{P}_r$ and $\tau \in \mathcal{P}_s$. Let $\mu \in \mathcal{P}_{r+s}$ be the permutation defined by $\mu(\mathcal{I}, \mathcal{J}) = (\sigma(\mathcal{I}), \mu(\mathcal{J}))$. Show that $\mathbf{sgn}(\mu) = \mathbf{sgn}(\sigma)\,\mathbf{sgn}(\tau)$.

6.6 Anti-symmetric tensors

In this section we consider probably the most important subspace of $\otimes^n \mathbb{R}^m$. We define the anti-symmetric tensor product (sometimes called the wedge product) of $\mathbf{x}_i$ to be

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma) \otimes_{i=1}^n \mathbf{x}_{\sigma(i)}.$$

We will use the convenient notation $\wedge_{i=1}^n \mathbf{x}_i$ for the left hand side of the above equation. We will denote the span of all wedge products of n vectors from $\mathbb{R}^m$ as $\wedge^n \mathbb{R}^m$.

Exercise 359 Show that

$$\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_i \wedge \cdots \wedge \mathbf{x}_j \wedge \cdots \wedge \mathbf{x}_n = (-1)\, \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_j \wedge \cdots \wedge \mathbf{x}_i \wedge \cdots \wedge \mathbf{x}_n.$$

We will write this fact succinctly as $\wedge_{i=1}^n \mathbf{x}_i = \mathbf{sgn}(\sigma) \wedge_{i=1}^n \mathbf{x}_{\sigma(i)}$ for any permutation $\sigma \in \mathcal{P}_n$. (Prove it.)

Exercise 360 Give an example of $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$ such that

$$(\mathbf{x} \wedge \mathbf{y}) \otimes \mathbf{z} - \mathbf{z} \otimes (\mathbf{x} \wedge \mathbf{y}) \neq c(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}),$$

for any scalar c. This shows that a naive definition of anti-symmetric tensor product is not associative.

Let $H_{m,n} = \{(i_1, i_2, \ldots, i_n) \,|\, 1 \leq i_k < i_{k+1} \leq m\}$. That is, $H_{m,n}$ is the set of n-tuples with strictly increasing components with values restricted to the integers $1, \ldots, m$. We claim that set of anti-symmetric tensors

$$\mathbf{h}_{\mathcal{I}} = \sqrt{n!} \wedge_{i=1}^n \mathbf{e}_{\mathcal{I}_i}, \qquad \mathcal{I} \in H_{m,n},$$

is an orthonormal basis for $\wedge^n \mathbb{R}^m$.

Exercise 361 Show that $\mathbf{h}_{\mathcal{I}}^T \mathbf{h}_{\mathcal{J}} = 0$ for $\mathcal{I}, \mathcal{J} \in H_{m,n}$ and $\mathcal{I} \neq \mathcal{J}$.

Exercise 362 Show that $\mathbf{h}_{\mathcal{I}}^T \mathbf{h}_{\mathcal{I}} = 1$ for $\mathcal{I} \in H_{m,n}$.

Exercise 363 Show that if $\mathcal{I} \in G_{m,n}$, and $\mathcal{I} \notin H_{m,n}$, then $\wedge_i \mathbf{e}_{\mathcal{I}_i} = \mathbf{0}$.

So we just need to show that $\mathbf{h}_{\mathcal{I}}$ spans $\wedge^n \mathbb{R}^m$. To do that it is sufficient to check that all elementray anti-symmetric tensors $\wedge_i \mathbf{x}_i$, are in the span. We calculate the linear combination as follows

$$\begin{aligned}
\wedge_{i=1}^n \mathbf{x}_i &= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\otimes_{i=1}^n \mathbf{x}_{\sigma(i)}\\
&= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\otimes_{i=1}^n \sum_{k=1}^m \mathbf{e}_k x_{k,\sigma(i)},\\
&= \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\sum_{\mathcal{I}\in F_{m,n}}\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} x_{\mathcal{I}_i,\sigma(i)}\\
&= \frac{1}{n!}\sum_{\mathcal{I}\in F_{m,n}}\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} x_{\mathcal{I}_i,\sigma(i)}\\
&= \frac{1}{n!}\sum_{\mathcal{I}\in F_{m,n}}\left(\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}\right)\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i}.
\end{aligned}$$

Now we observe that for each $\mathcal{J}\in F_{m,n}$ there is a $\mathcal{I}\in G_{m,n}$ and a $\tau\in\mathcal{P}_n$ (though the τ may not be unique) such that $\mathcal{J}=\tau(\mathcal{I})$.

Exercise 364 Show that for such a pair

$$\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{J}_i,\sigma(i)} = \mathbf{sgn}(\tau)\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}.$$

Therefore we can further group the terms together and obtain

$$\begin{aligned}
\wedge_{i=1}^n \mathbf{x}_i &= \frac{1}{n!}\sum_{\mathcal{I}\in F_{m,n}}\left(\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}\right)\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i}\\
&= \sum_{\mathcal{I}\in G_{m,n}}\frac{1}{n!}\left(\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}\right)\frac{1}{\mathcal{I}!}\sum_{\tau\in\mathcal{P}_n}\mathbf{sgn}(\tau)\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i}\\
&= \sum_{\mathcal{I}\in G_{m,n}}\left(\frac{1}{\mathcal{I}!}\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}\right)\wedge_{i=1}^n \mathbf{e}_{\mathcal{I}_i}\\
&= \sum_{\mathcal{I}\in H_{m,n}}\left(\sum_{\sigma\in\mathcal{P}_n}\mathbf{sgn}(\sigma)\Pi_{i=1}^n x_{\mathcal{I}_i,\sigma(i)}\right)\wedge_{i=1}^n \mathbf{e}_{\mathcal{I}_i},
\end{aligned}$$

which proves our claim.

Therefore $\mathbf{dim}(\wedge^n\mathbb{R}^m)$ is the cardinality of the set $H_{m,n}$ which gives easily

$$\mathbf{dim}(\wedge^n\mathbb{R}^m) = \binom{m}{n}.$$

In particular $\wedge^n\mathbb{R}^m = \{\mathbf{0}\}$ if $n > m$, and $\mathbf{dim}(\wedge^m\mathbb{R}^m) = 1$. Also note that $\mathbf{dim}(\wedge^n\mathbb{R}^m) = \mathbf{dim}(\wedge^{m-n}\mathbb{R}^m)$.

Determinant Let $\mathbf{A}$ be an $n \times n$ matrix. Its determinant is defined to be **110**

$$\det(\mathbf{A}) = \sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma)\Pi_{i=1}^n \mathbf{A}_{i,\sigma(i)}.$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two $m \times n$ matrices. We will use the notation $\mathbf{X}_i$ to denote column i of $\mathbf{X}$.

Exercise 365 Show that

$$(\wedge_{i=1}^n \mathbf{X}_i)^T(\wedge_{i=1}^n \mathbf{Y}_i) = \frac{1}{n!}\det(\mathbf{X}^T\mathbf{Y}).$$

Exercise 366 Show that

$$|\det(\mathbf{X}^T\mathbf{Y})| \leq \sqrt{\det(\mathbf{X}^T\mathbf{X})\det(\mathbf{Y}^T\mathbf{Y})}.$$

Example 8 At this stage it is good to do the following exercise from Bhatia. Note that $\mathbf{dim}(\otimes^3\mathbb{R}^3) = 27$, $\mathbf{dim}(\vee^3\mathbb{R}^3) = 10$ and $\mathbf{dim}(\wedge^3\mathbb{R}^3) = 1$. Find an element of $(\vee^3\mathbb{R}^3 \oplus \wedge^3\mathbb{R}^3)^\perp$. A brute force approach that will work is to pick a random vector in $\otimes^3\mathbb{R}^3$ and orthogonalize it against all suitable $\mathbf{g}_\mathcal{I}$ and $\mathbf{h}_\mathcal{I}$. A simpler way is to proceed as follows. Observe that every vector in $\wedge^3\mathbb{R}^3$ is a linear multiple of $\wedge_{i=1}^3\mathbf{e}_i$. Motivated by this consider the vector $\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1$. Clearly it is orthogonal to $\wedge^3\mathbb{R}^3$. In $\vee^3\mathbb{R}^3$ it is clearly orthogonal to all $\mathbf{g}_\mathcal{I}$ except possibly for $\mathbf{g}_{(1,1,2)}$. A quick check shows that it is orthogonal to this one too.

As in the symmetric case calculations become easier to do if we can define a fully associative wedge product (also called the Grassmann product). Like before we need to find the orthogonal projector $P_\wedge$, onto $\wedge^n\mathbb{R}^m$. We define it on the canonical basis vectors as follows

$$P_\wedge(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i}) = \wedge_{i=1}^n \mathbf{e}_{\mathcal{I}_i}, \qquad \text{for } \mathcal{I} \in F_{m,n}.$$

We need to check if this is indeed an orthogonal projector. We begin by checking it is idempotent. It is sufficient to check this on $\mathbf{h}_\mathcal{I}$ for $\mathcal{I} \in H_{m,n}$.

$$\begin{aligned} P_\wedge \mathbf{h}_\mathcal{I} &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma) P_\wedge \left(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right) \\ &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma) \wedge_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}} \\ &= \mathbf{h}_\mathcal{I}. \end{aligned}$$

Finally we check if $\mathbf{x} - P_\wedge \mathbf{x}$ is perpendicular to $P_\wedge \mathbf{x}$ for all vectors $\mathbf{x}$. It is sufficient to check $\otimes_i \mathbf{e}_{\mathcal{I}_i} - P_\wedge(\otimes_i \mathbf{e}_{\mathcal{I}_i})$ is perpendicular to all $\mathbf{h}_{\mathcal{J}}$. It is clear that if $\sigma(\mathcal{I}) \in G_{m,n}$, but $\sigma(\mathcal{I}) \notin H_{m,n}$, for some permutation σ, then clearly the orthogonality condition holds. So we only need to check when $\mathcal{I} \in H_{m,n}$. Thus for $\mathcal{I}, \mathcal{J} \in H_{m,n}$ we must compute

$$\left(\sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma) \otimes_{i=1}^n \mathbf{e}_{\mathcal{J}_{\sigma(i)}}\right)^T (\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} - P_\wedge(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i})).$$

Clearly if $\mathcal{I} \neq \mathcal{J}$ the above inner product is zero. Thus we only need to check when $\mathcal{I} = \mathcal{J} \in H_{m,n}$.

$$\left(\sum_{\sigma \in \mathcal{P}_n} \mathbf{sgn}(\sigma) \otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_{\sigma(i)}}\right)^T (\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i} - P_\wedge(\otimes_{i=1}^n \mathbf{e}_{\mathcal{I}_i})) = 1 - \frac{n!}{n!},$$

which confirms that $P_\wedge$ is the orthogonal projector onto $\wedge^n \mathbb{R}^m$.

Next, for $\mathcal{I} \in H_{m,n_1}$ and $\mathcal{J} \in H_{m,n_2}$ we compute the anti-symmetric tensor

$$\begin{aligned}
&P_\wedge \left(\left(\frac{1}{n_1!} \sum_{\sigma \in \mathcal{P}_{n_1}} \mathbf{sgn}(\sigma) \otimes_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_{\sigma(i)}} \right) \otimes \left(\frac{1}{n_2!} \sum_{\sigma \in \mathcal{P}_{n_2}} \mathbf{sgn}(\sigma) \otimes_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_{\sigma(i)}} \right) \right) \\
&= \frac{1}{n_1! n_2!} \sum_{\substack{\sigma \in \mathcal{P}_{n_1} \\ \tau \in \mathcal{P}_{n_2}}} \mathbf{sgn}(\sigma)\, \mathbf{sgn}(\tau) P_\wedge \left((\otimes_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_{\sigma(i)}}) \otimes (\otimes_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_{\sigma(i)}}) \right) \\
&= \frac{1}{n_1! n_2!} \sum_{\substack{\sigma \in \mathcal{P}_{n_1} \\ \tau \in \mathcal{P}_{n_2}}} \mathbf{sgn}(\sigma)\, \mathbf{sgn}(\tau) (\wedge_{i \in (\sigma(\mathcal{I}), \tau(\mathcal{J}))} \mathbf{e}_i) \\
&= \frac{n_1! n_2!}{n_1! n_2!} \wedge_{i \in (\mathcal{I}, \mathcal{J})} \mathbf{e}_i.
\end{aligned}$$

Therefore we can extend the definition of wedge product to anti-symmetric tensors by first defining it on the canonical basis for $\wedge^n \mathbb{R}^m$:

$$(\wedge_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}) \wedge (\wedge_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_i}) = \wedge_{i=1}^{n_1+n_2} \mathbf{e}_{(\mathcal{I},\mathcal{J})_i}.$$

We then extend it by linearity in each argument. Therefore for $\mathbf{x} \in \wedge^{n_1} \mathbb{R}^m$ and $\mathbf{y} \in \wedge^{n_2} \mathbb{R}^m$, since

$$\mathbf{x} = \sum_{\mathcal{I} \in H_{m,n_1}} x_{\mathcal{I}} \wedge_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}, \qquad \text{and} \qquad \mathbf{y} = \sum_{\mathcal{I} \in H_{m,n_2}} y_{\mathcal{I}} \wedge_{i=1}^{n_2} \mathbf{e}_{\mathcal{I}_i},$$

we have

$$\mathbf{x} \wedge \mathbf{y} = P_\wedge(\mathbf{x} \otimes \mathbf{y}) = \sum_{\substack{\mathcal{I} \in H_{m,n_1} \\ \mathcal{J} \in H_{m,n_2}}} x_\mathcal{I} y_\mathcal{J} \wedge_{i=1}^{n_1+n_2} \mathbf{e}_{(\mathcal{I},\mathcal{J})_i}.$$

Note that many terms on the right-hand side can be zero. Furthemore observe that for $\mathcal{I} \in H_{m,n_1}$ and $\mathcal{J} \in H_{m,n_2}$

$$(\wedge_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}) \wedge (\wedge_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_i}) = (-1)^{n_1 n_2} (\wedge_{i=1}^{n_2} \mathbf{e}_{\mathcal{J}_i}) \wedge (\wedge_{i=1}^{n_1} \mathbf{e}_{\mathcal{I}_i}).$$

Exercise 367 Show that for anti-symmetric tensors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ and scalar α

- $(\mathbf{x} + \alpha\mathbf{y}) \wedge \mathbf{z} = \mathbf{x} \wedge \mathbf{z} + \alpha\mathbf{y} \wedge \mathbf{z}$,
- $(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} = \mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})$,
- $\mathbf{x} \wedge \mathbf{y} = (-1)^{n_1 n_2} \mathbf{y} \wedge \mathbf{x}$, if $\mathbf{x} \in \wedge^{n_1}\mathbb{R}^m$ and $\mathbf{y} \in \wedge^{n_2}\mathbb{R}^m$.

Exercise 368 Show that if $\mathbf{v}_i \in \mathbb{R}^m$ then $\wedge_{i=1}^{n} \mathbf{v}_i = 0$ iff the $\mathbf{v}_i$ are linearly dependent.

6.7 Anti-symmetric tensor powers