Home Work 1.

Two pages.

Reading assignments don't have to be turned in. You can choose to do the problems in this homework **or** the exercises in sections 2.1–2.5 of my notes posted on the class web-site.

1. **Reading assignment.** Read chapters 1, 2 and 3 of Dym's book.

2. **Reading assignment.** Read chapter 2 of the notes posted on the class web-site.

3. Find a way to represent complex numbers as 2×2 **real** matrices, such that arithmetic operations ($+$, $-$, $\times$, and $\div$) on complex numbers becomes equivalent to arithmetic operations on their matrix representations instead. That is, if $T(z)$ is the 2×2 real matrix representing the complex number z, then

$$\begin{aligned} T(z_1 + z_2) &= T(z_1) + T(z_2), \\ T(z_1 - z_2) &= T(z_1) - T(z_2), \\ T(z_1 z_2) &= T(z_1)T(z_2), \\ T(z_1/z_2) &= (T(z_2))^{-1}T(z_1), \end{aligned}$$

for all complex numbers z_1 and z_2. *Hint:* The entries of $T(z)$ will depend on the real and imaginary parts of z. Look at the expression for the product of two complex numbers and two 2×2 matrices.

4. A. Show that the **exchange** operator

$$E_{ij}x = E_{ij}\begin{pmatrix} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \end{pmatrix}$$

which exchanges the i-th and j-th components of a vector leaving all others untouched, is a linear operator

B. Find a matrix representation for E_{ij}.

C. Show that the **permutation** operator

$$P_{\sigma_1\sigma_2\cdots\sigma_n}\, x = \begin{pmatrix} x_{\sigma_1} \\ x_{\sigma_2} \\ \vdots \\ x_{\sigma_n} \end{pmatrix}, \quad 1 \leq \sigma_i \leq n, \quad \sigma_i \neq \sigma_j \text{ if } i \neq j,$$

is a linear operator and find its matrix representation.

D. How many distinct exchange matrices are there of size n?

E. How many distinct permutation matrices are there of size n?

F. Show that every $n \times n$ permutation matrix can be written as the product of *at most* n exchange matrices.

5. Let P_n denote the set of all poynomials of degree at most n with real coefficients.

   A. Find a way to represent any polynomial in P_n as a real column matrix with $n+1$ elements. That is, find an invertible map T from P_n to $\mathcal{R}^n$ such that $T(p_1 + p_2) = T(p_1) + T(p_2)$, and $T(\alpha p_1) = \alpha T(p_1)$, for any two polynomials p_1 and p_2 from P_n, and any scalar α.

   B. Can you find a representation that is different from the one in the previous part? There are infinitely many.

   C. Find an $(n+1) \times (n+1)$ matrix D, such that $DT(p) = T(p')$, for any polynomial p in P_n, where p' denotes the derivative of p.

   D. Is the matrix D invertible? Left-invertible? Right-invertible?