Home Work 3

Due on October 20, 2008
Two pages

Reading assignments don't have to be turned in.

1. **Reading assignment.** Read chapters 7, 8, 9, 10 and 11 of Dym's book. The lectures in the next few weeks will concern these chapters.

2. **Reading assignment.** Read chapters 2, 3 and 4 of the class notes posted on the class web-site. Some typos have been fixed and new material has been added, so be sure to download the updated notes.

3. Show (in complete detail) that X is a full column rank matrix if and only if $X^T X$ is non-singular (invertible). Assume X is a real matrix.

4. Show how to construct at least one left-inverse for a full column rank matrix, and one right-inverse for a full row rank matrix.

5. If X and Y are full column rank matrices of rank p, show that the rank of XY^T is equal to p.

6. Show that if A is a rank p matrix then you can find two full column rank matrices X and Y of rank p such that $A = XY^T$.

7. If X and Y are two full column rank matrices of rank p, find *all* matrices A for which $R(X) = R(A)$ and $R(A^T) = R(Y)$.

8. Let X and Y be two full column rank real matrices. What are the conditions (if any) on X and Y such that there exsists a real matrix A such that $AX = Y$ and $Y^T A = X^T$? Find all such A when the conditions are satisfied.

9. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function that satisfies the following conditions

   – $f(v) \geq 0$ for all $v \in \mathbb{R}^n$

   – $f(v) = 0$ iff $v = 0$

   – $f(\alpha v) = |\alpha| f(v)$ for all $\alpha \in \mathbb{R}$ and all $v \in \mathbb{R}^n$

   – The set $\{v : f(v) \leq 1\}$ is convex

   then f defines a norm on $\mathbb{R}^n$.

10. Show that for $p \geq 1$ and $p^{-1} + q^{-1} = 1$,

$$\|x\|_p = \max_{y \neq 0} \frac{|y^T x|}{\|y\|_q}, \qquad x \in \mathbb{R}^n.$$

Hint: You can use Hölder's inequality for part of the proof.

11. Show that the Frobenius norm is sub-multiplicative. *Hint*: See Exercise 122 of the notes (October 13, 2008).

12. Let $\|A^{-1}\|\|E\| < 1$ for some induced matrix norm. Show that $A + E$ is non-singular and that

$$\frac{\|(A+E)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \|A\|\|A^{-1}\|\frac{\|E\|}{\|A\|}\frac{1}{1 - \|A^{-1}\|\|E\|}.$$

Hint: See Exercise 151 of the notes (October 13, 2008).

13. Show that $\|A\|_2^2 \leq \|A\|_1\|A\|_\infty$. *Hint*: See Exercise 125 of the notes (October 13, 2008).