Home Work 4

Two pages.
Due on October 30, 2008

Reading assignments don't have to be turned in.

1. **Reading assignment.** Read chapters 7, 8, 9, 10 and 11 of Dym's book.

2. **Reading assignment.** In the next few lectures we will cover chapters 4 and 6 in Dym's book.

3. We say that a sequence $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$, of n-dimensional vectors **converges** to x if

$$\lim_{k\to\infty} \|x^{(k)} - x\| = 0.$$

   Show that in $\mathbb{R}^n$, $x^{(i)} \to x$ if and only if $x_k^{(i)} \to x_k$ for k from 1 to n.

4. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if $\text{rank}(A) = n$, then $\|x\|_A \equiv \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

5. Prove or disprove

$$v \in \mathbb{R}^n \Rightarrow \|v\|_1\|v\|_\infty \le \frac{1+\sqrt{n}}{2}\|v\|_2^2.$$

6. Let B be any submatrix of A. Show that $\|B\|_p \le \|A\|_p$.

7. Show that if $s \in \mathbb{R}^n$ and $s \neq 0$, and $E \in \mathbb{R}^{n\times n}$, then

$$\left\| E\left(I - \frac{ss^T}{s^Ts}\right)\right\|_F^2 = \|E\|_F^2 - \frac{\|Es\|_2^2}{s^Ts}.$$

8. Suppose $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Show that if $E = uv^T$ then $\|E\|_F = \|E\|_2 = \|u\|_2\|v\|_2$ and that $\|E\|_\infty = \|u\|_\infty\|v\|_1$.

9. Suppose $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and $0 \neq s \in \mathbb{R}^n$. Show that $E = (y - As)s^T/s^Ts$ has the smallest 2-norm of all $m \times n$ matrices E that satisfy $(A+E)s = y$.

10. Let X and Y be two full column rank matrices. We know from the last homework that there are many matrices A which satisfy the pair of equations

$$AX = Y,$$
$$Y^T A = X^T.$$

From the set of all such matrices A find the one that has the *least* Frobenius norm.

11. Let M be a real symmetric $n \times n$ matrix, and let L_M be a linear operator from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$, defined by the equation

$$L_M(A) = \frac{A^T M + MA}{2}.$$

Define the matrix (or operator) 2-norm of L_M (denoted by $\|L_M\|_2$) by

$$\|L_M\|_2 = \max_{\|A\|_F=1} \|L_M(A)\|_F.$$

Show that

$$\|L_M\|_2 \leq \|M\|_F.$$

Extra credit: Show that $\|L_M\|_2 = \|M\|_2$.