Homework 5

1. Show that the mean kinetic energy of a three-dimensional Fermi gas of *N* free electrons at 0 K is $U_0 = \frac{3}{5} N U_F$

2. Classical limit of Fermi gas. (a) For the Fermi gas of electrons and three-dim density-of-states derived in class, show that the mean specific energy of a Fermi gas is given by $(3/2)\rho k_B T$ where ρ is the electron density (clue: take the high-temp approximation of the Fermi-Dirac function).
   (b) Derive an expression for the Fermi energy $U_F(T)$ in the high-temp limit in terms of ρ, T, m*, and fundamental constants. For what electron density does $U_F = 0$ at T = 300 K, assuming m* = m_e the mass of an electron in vacuum ? If the temperature is increased at this same density, what happens to the Fermi energy ? What does this mean physically ?
   (c) What is the specific heat capacity of the electron gas in the high-temp approximation. How does this compare to high-temperature heat capacity from acoustical phonons in any crystal assuming one electron is contributed to the gas by each primitive cell in the crystal ?.

3. Show that the Fermi energy of a Fermi gas in two dimensions is given by $U_F(T) = k_B T \ln\left[\exp\left(\pi\rho_S \hbar^2 / m^* k_B T\right) - 1\right]$ for ρ_S electrons per unit area (sheet chart density).

4. Numerical integration to evaluate Fermi-Dirac integrals.
   Some of the most useful definite integrals in all of electronics are the Fermi integral of order y, defined by $F_y(x_F) = \int_0^\infty \frac{x^y dx}{\exp(x - x_F) + 1}$. $F_{1/2}(x_F)$ comes up in the relationship between density and U_F, and $F_{3/2}(x_F)$ comes in the mean energy. The Fermi integrals cannot be reduced to closed form, so look-up tables were used by scientists and engineers for a long time. Today these integrals can be integrated quickly and with high accuracy on PCs and laptops by a number of different computational tools, such as Matlab. Use your favorite computational tool to numerically evaluate $F_{1/2}(x_F)$ for x_F = -20 to x_F = 100 in increments of x_F = 10. Use these values to plot the electron concentration in a Fermi gas for m* = m_e and T = 300 K.
   (clue for Matlab users: consider the following two lines in the workspace:
   f = inline('x.^0.5/(exp(x-xf)+1)'); g = quadl(f,0,xmax,1e-6,0,xf)
   where xf is the parametric normalized Fermi energy, and xmax is the maximum range of the integral, which should be set to about 10 xf or more. (quadl is a nice integration algorithm… please give it a try if you have access to MATLAB).

5. The reaction of the Fermi gas to an internal slowly-varying electrostatic-potential disturbance $\phi(x)$ was derived in class in the non-degenerate limit and involved a length scale, the Debye screening length $L_D = (\varepsilon k_B T/e^2 n)^{1/2}$ where $\varepsilon = \varepsilon_r \varepsilon_0$ is the dielectric constant and n is the density free electrons.
   (a) Express L_D in terms of the circular plasma frequency and the rms velocity (i.e., variance) for a Maxwellian distribution.
   (b) A similar but tedious calculation in the degenerate limit results in a Thomas-Fermi screening length, $L_{TF} = (2\varepsilon U_F/3ne^2)^{1/2}$. Re-write this in terms of the circular plasma frequency and the Fermi velocity, and contrast to the answer in (a).
   (c) Now evaluate the following quantities for two very important materials, GaAs and Cu, in the stated MKSA units and at 300 K:
   RMS thermal velocity, Fermi energy, Fermi velocity, circular plasma frequency, linear plasma frequency, Debye screening length, and Thomas-Fermi screening length.

   (for GaAs assume $n = 2.0 \times 10^{18}\ cm^{-3}$, $\varepsilon_r = 12.8$, $m^* = 0.067\ m_e$;
   for Cu assume $n = 8.4 \times 10^{22}\ cm^{-3}$, $e_r = 1$, and $m^* = m_e$)