Homework 4 Solutions

1) **Hall Effect**

(a) Newton's equation with scattering (relaxation time approximation)

$$m\dot{\vec{\upsilon}} + m\vec{\upsilon}/\tau = \vec{F} = q\left(\vec{E} + \vec{\upsilon} + \vec{B}\right)$$

$$\vec{\upsilon} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \upsilon_x & \upsilon_y & \upsilon_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{matrix} \hat{x}\left(\upsilon_y B_z - v_z B_y\right) \\ +\hat{y}\left(v_Z B_x - v_X B_z\right) \\ +\hat{z}\left(\upsilon_x B_y - \upsilon_y B_x\right) \end{matrix}$$

In steady state, $m(d\vec{v}/dt) = 0$, so that

$$\upsilon_x = \frac{q\tau}{m}\left(E_x + \upsilon_y B_z - \upsilon_z B_y\right); \quad \upsilon_y = \frac{q\tau}{m}\left(E_y + \upsilon_z B_x - \upsilon_x B_z\right);$$

$$\upsilon_z = \frac{q\tau}{m}\left(E_z + \upsilon_x B_y - \upsilon_y B_x\right)$$

In the special geometry of the Hall sample, $B_z = B_0; B_x = B_y = 0$. In kinetic theory

$J_x = nq\upsilon_x, J_y = nq\upsilon_y, J_z = nq\upsilon_z$ (current density components) where n is the carrier density.

Therefore, in the Hall geometry, $J_y = 0 \Rightarrow \upsilon_y = 0, J_Z = 0 \Rightarrow \upsilon_z = 0$, and we get

$$\upsilon_x = \frac{q\tau}{m}\left(E_x\right) \tag{1}$$

$$0 = \frac{q\tau}{m}\left(E_y - \upsilon_x B_0\right) \tag{2}$$

$$0 = \frac{q\tau}{m}\left(E_z\right) \tag{3}$$

$$(2) \Rightarrow E_y = \upsilon_x B = \frac{q\tau}{m} E_x B_0 \Rightarrow V_y = L_y \frac{q\tau V_x}{mL_x} B_0 \tag{4}$$

(b) From (1)

$$E_x = \frac{m\upsilon_x}{q\tau} = \frac{mJ_x}{q\tau nq}. \tag{5}$$

So we can write $$V_y = L_y \frac{q\tau}{m}\frac{mJ_xB}{q\tau nq} = \frac{I_xB_0}{L_z nq}.$$

(c) For the specific geometry $L_x = 1$ cm, $L_y = L_z = 1$ mm, and $V_y = +5$ mV, as defined in the figure so that the electric potential decreases in going from small y to larger y $\Rightarrow E_y = -d\phi/dy$ is positive. Since E_y and E_x are both positive, we must take positive sign in equation (4), so that the carriers must have a positive charge (e.g., holes). For the geometry at hand and assuming uniform fields,

$$V_x = E_x L_x \; or \; E_x = 1 \;\; V/cm \qquad V_y = E_y L_y or \; E_y = 0.05 \;\; V/cm$$

$$\therefore \mu \approx \frac{e\tau}{m} = \frac{E_y/E_x}{B} = \frac{0.05}{0.1} = 0.5 \;\; m^2/V-s = 5000 \;\; cm^2/V-s$$

2. Magnetoconductivity in Three Dimensions

In class we derived the following set of equations for kinetic motion of charge carriers in magnetic and electric field along z axis.

$$\upsilon_x = \left(\mu E_x + \omega_c\tau\upsilon_y\right)\frac{q}{e} \;(1)\,; \quad \upsilon_y = \left(\mu E_y - \omega_c\tau\upsilon_x\right)\frac{q}{e} \;(2); \quad \upsilon_z = \left(\mu E_z\right)\frac{q}{e} \;(3)$$

where $\omega_c = \dfrac{qB}{m}$ is the cyclotron resonance (circular) frequency.

We use (1) and (2) to solve uniquely for υ_x and υ_y.

$$(2) \to (1) \Rightarrow \upsilon_x = \frac{q}{e}\left(\mu E_x\right) + \omega_c\tau\left[\mu E_y - \omega_c\tau\upsilon_x\right]\cancelto{1}{\frac{q^2}{e}}$$

or $$\upsilon_x\left[1 + \left(\omega_c\tau\right)^2\right] = \frac{q}{e}\mu E_x + \omega_c\tau\mu E_y$$

$$J_x = nq\upsilon_x = \frac{n\cancel{\frac{q^2}{e\mu}}E_x + \omega_c\tau nq\mu E_y}{e \qquad 1+\left(\omega_c\tau\right)^2}$$

but $\sigma_0 \equiv ne\mu$ (dc conductivity) so

$$J_x = \frac{\sigma_0}{1+\left(\omega_c\tau\right)^2}\left(E_x + \omega_c\tau E_y\left(\frac{q}{e}\right)\right) \qquad (4)$$

$$(1) \to (2) \Rightarrow \upsilon_y = \frac{q}{e}\mu E_y - \omega_c \tau \left[\mu E_x + \omega_c \tau \upsilon_y \right] \cancel{\frac{q^2}{e^2}}_{1}$$

or
$$\upsilon_y = \left[1 + (\omega_c \tau)^2 \right] = \frac{q}{e}\mu E_y - \omega_c \tau \mu E_x$$

$$J_y = nq\upsilon_y = \frac{\sigma_0}{1+(\omega_c\tau)^2}\left(-\omega_c \tau E_x \frac{q}{e} + E_y \right) \tag{5}$$

$$(3) \Rightarrow J_z = nq\upsilon_z = \left(\frac{q}{e}\right) nq\mu E_z \equiv \sigma_0 \frac{1+(\omega_c\tau)^2}{1+(\omega_c\tau)^2} E_z \tag{6}$$

We can combine (4), (5), and (6) in matrix form as

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \frac{\sigma_0}{1+(\omega_c\tau)^2} \begin{pmatrix} 1 & \frac{q}{e}\omega_c\tau & 0 \\ -\frac{q}{e}\omega_c\tau & 1 & 0 \\ 0 & 0 & 1+(\omega_c\tau)^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

In the high magnetic-field limit $\omega_c\tau >> 1$,

$$\sigma_{yx} = \frac{-\frac{q}{e}\sigma_0\omega_C\tau}{1+(\omega_c\tau)^2} \to \frac{-\frac{q}{e}\sigma_0}{\omega_c\tau} = \frac{-qmne^2\tau}{emeB\tau} = \frac{-nq}{B} \to \frac{ne}{B} \quad \text{for electrons}$$

$$\sigma_{xy} = \frac{(q/e)\sigma_0\omega_c\tau}{1+(\omega_c\tau)^2} \to \frac{(q/e)\sigma_0}{\omega_C\tau} = \frac{nq}{B} \to \frac{-ne}{B} \quad \text{for electrons}$$

In the same limit:

$$\sigma_{xx} = \frac{\sigma_0}{1+(\omega_c\tau)^2} \to \frac{\sigma_0}{(\omega_c\tau)^2} \to \frac{ne^2\tau m^2}{(eB)^2\tau^2 m}$$

or $$\sigma_{xx} = \frac{nm}{\tau B^2} = \frac{qn}{\omega_c \tau B} \to 0$$ in high B-field limit

3. Joule Heating

(a) From basic assumptions of kinetic theory, collisions are randomizing and leave the scattered particle with a mean velocity appropriate to the temperature around the scattering center. Hence if consecutive scattering events occur close enough that the temperature is the same, then all of the kinetic energy gained between collisions is transferred to the ions upon the second collision. In the absence of an electric field, the velocity vector emerging from first collision can be written as the isotropic velocity

$$\vec{\upsilon} = \upsilon_0 \hat{r}_0 = \upsilon_0 \left(\hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta \right)$$

where θ,ϕ are the polar and azimuthal angles in spherical coordinates..

In the presence of an electric field along the z axis, the z component of velocity has a term that increases linearly with time (solution to Newton's equation).

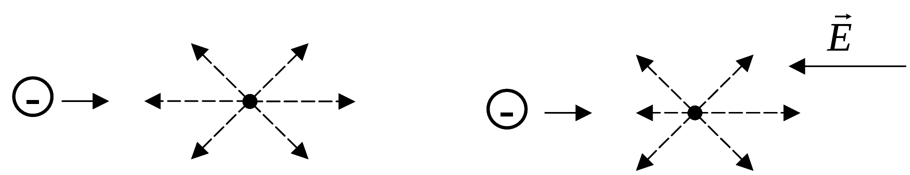

Fig. 1.

$$\upsilon_z = \upsilon_0 + \frac{qE_0 t}{m}\ ;\ \vec{E} = E_0 \hat{z}$$

So, $$\vec{\upsilon} = \upsilon_0 \sin\theta \cos\phi \cdot \hat{x} + \upsilon_0 \sin\theta \sin\phi \cdot \hat{y} + \left(\upsilon_0 \cos\theta + \frac{qE_0 t}{m} \right) \cdot \hat{z} \tag{1}$$

To calculate how much energy is gained between collisions, we first have to spatially average over all possible angles (overbar denotes spatial average):

$$\bar{\vec{\upsilon}} = \frac{\int \vec{\upsilon} d\Omega}{\int d\Omega} = \int_0^{2\pi} \int_0^{\pi} \frac{\vec{\upsilon} \sin\theta d\theta d\phi}{4\pi}$$

The first (zero field term) from (1) yields

$$\bar{\vec{\upsilon}}_1 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\upsilon_0 \sin\theta \cos\phi \cdot \hat{x} + \upsilon_0 \sin\theta \sin\phi \cdot \hat{y} + \upsilon_0 \cos\theta \cdot \hat{z} \right) \sin\theta d\theta d\phi = 0$$

The second term (electric field term) is

$$\bar{\bar{\upsilon}}_2 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{qE_0t}{m}\right)\hat{z}\sin\theta d\theta d\phi = \frac{\hat{z}}{2}\frac{qE_0t}{m}\left(-\cos\theta\right)\Bigg|_0^{\pi} = \hat{z}\frac{qE_0t}{m}$$

$$U_K^* = \frac{1}{2}m\left|\bar{\bar{\upsilon}}\right|_2^2 = \frac{m}{2}\left(\frac{qE_0t}{m}\right)^2 = \frac{1}{2}\frac{\left(qE_0t\right)^2}{m}$$

where U^*_K denotes the average single-particle kinetic energy and the | | denotes the vector magnitude operation.

(b) We can combine this space-averaged kinetic energy with the normalized probability, $P\left(t+dt\right)$ that a collision occurs between t and t + dt : $e^{-t/\tau}dt/\tau$:

$$<U_K^*> = \int_0^{\infty} U_K^* P\left(t, dt\right)dt = \int_0^{\infty}\frac{1}{2}\frac{qE_0t^2}{m}\frac{e^{-t/\tau}dt}{\tau}$$

$$<U_K^*> = \left(\frac{qE_0t}{m}\right)^2\frac{1}{2\tau}\int_0^{\infty}t^2e^{-t/\tau}dt$$

We integrate $\int_0^{\infty}t^2e^{-t/\tau}dt$ by parts twice (or look in integral tables) and we find $\int_0^{\infty}t^2e^{-t/\tau}dt = 2\tau^3$. So

$$<U_K^*> = \left(\frac{qE_0}{m}\right)^2\frac{2\tau^3}{2\tau} = \frac{(qE_0\tau)^2}{m}$$

which is the energy loss per electron per collision. It is useful to do a dimensional analysis:

$$\frac{Energy\ loss}{cm^3-\sec} = \frac{Energy\ loss}{electron-collision}\cdot\frac{electrons}{cm^3}\cdot\frac{collisions}{\sec}$$

$$\text{“} \qquad \text{“} \qquad = \frac{\left(qE_0\tau\right)^2}{m}\cdot n\cdot\frac{1}{\tau}$$

$$\text{“} \qquad \text{“} \qquad = \frac{nq^2\tau}{m}\cdot E_0^2 = \sigma E^2 \quad \text{Joule Heat}$$

To calculate the power loss in a wire length L and cross section A, we integrate over the volume

$$P_L = \int\sigma E^2 dV = \sigma E^2\int dV = \sigma E^2 L\cdot A \quad \text{(assuming E is uniform in wire)}$$

We can rewrite this as $P_L = \frac{\sigma^2}{\sigma}E^2L\frac{A^2}{A} = \left(\sigma EA\right)^2\left(\frac{L}{\sigma A}\right) = \left(J^2A^2\right)\left(\frac{\rho L}{A}\right)$

or

$$P_L = I^2R$$