Homework#4 Solutions

Exercise (1). Phonon Thermal Conductivity

(a) C' = $3nk_B$ where n is the density of atoms per unit volume. So

$$C' = 3nk_B = 3\frac{N}{V_c}k_B = 3\cdot\frac{2}{45\times10^{-30}}\cdot k_B = 1.84\times10^6 J/k$$

(b) $K \simeq C'\upsilon\frac{X}{3}$, $X \approx \upsilon\cdot\tau = 7\times10^{-9}m$ for υ = 7000 m/s, τ = 1 ps. So K = 30.0 W/m-k or 0.3 W/cm-k.

Exercise (2). Thermal Conductivity of Al

a) For Al,

$n_e = 18.06\times10^{22}\ /cm^3 = 18.06\times10^{28}\ /m^2$, $\sigma = 3.65\times10^5\ /\Omega\text{–}cm$ (Table 3, Chapter 6, Kittel)

so that, $\tau = \frac{m\sigma}{neq^2} = \frac{(9.11\times10^{-31})(3.65\times10^7)}{(18.06\times10^{28})(1.6\times10^{-19})^2}$ or $\tau \simeq 0.72\times10^{-14}s = 7.2\,fs$

b) $C_{el} = \frac{1}{2}\pi^2 Nk_B T/T_F = 91.2\ \ T\cdot V[J]$ (Eq. 36, Kittel Chapter 6)

for $T_F = 13.49\times10^4 K,\ V\rightarrow volume$

$$C_{el}'(specific\ \ heat\ \ capacity) \equiv \frac{C_{el}}{V} = 91.2\ \ T\ J/m^3 = 2.69\times10^4\ J/m^3\ @\ 295K$$

c) So, since $\upsilon = \upsilon_F = 2.02\times10^{28}\ cm/s = 2.02\times10^6\ m/s$ (Al at RT)

$$K = \frac{1}{3}C_{el}'\upsilon^2\tau \approx 2.63\times10^2\ w/m\text{–}K$$

An accepted value for Thermal Conductivity is found in Kittel Chpater 5, Table 1.

$K = 2.37\,W/cm\text{–}K = 237\,W/m\text{–}K$ for Al @ 300K. Our estimate using $\frac{1}{3}C_{el}'\upsilon^2\tau$ is too high by 1.13 times/ This is caused largely by the inaccuracy of the Fermi derivation used to derive heat capacity from $\Delta U = U(T) - U(0)$. In this derivation we approximated $\frac{dU(T)}{dT}$ by $\frac{d\Delta U(T)}{dT}$, which becomes increasingly inaccurate as T rises above zero K.

Exercise (3) Given $\tau(U) = AU^{-S}$ and Maxwell-Boltzmann statistics,

$$<\tau> = \frac{<v^2\tau(v)>}{<v^2>} = \frac{m}{3k_BT}\frac{\int v^2\tau(v) f_o(v) v^2 dv}{\int f_o(v) v^2 dv} = \frac{2}{3k_BT}\frac{\int U^{3/2}\tau(U) f_o(U) dU}{\int U^{1/2} f_o(U) dU}$$

where $f_o(u) = n(\frac{h^2}{2\pi m k_BT})^{3/2} e^{-U/k_BT}$

$$\text{So } <\tau> = \frac{2}{3k_BT}\int_0^\infty \frac{u^{3/2}Au^{-s}e^{-U/k_BT}dU}{\int_0^\infty u^{1/2}e^{-U/k_BT}dU}$$

$$U' = U/k_BT \Rightarrow dU = k_BTdU'$$

$$<\tau> = \frac{2}{3k_BT}\frac{(k_BT)^{5/2}(k_BT)^{-s}\int_0^\infty AU'^{3/2-S}e^{-U'}dU'}{(k_BT)^{3/2}\int_0^\infty (U')^{1/2}e^{-U'}dU'}$$

Now recall definition $\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx$.

$$\text{So } <\tau> = \frac{2A}{3k_BT}(k_BT)^{1-S}\frac{\Gamma(5/2 - S)}{\sqrt{\pi}/2} = \frac{4A}{3\sqrt{\pi}(k_BT)^S}\Gamma(5/2 - S)$$

$$= \frac{4A}{3\sqrt{\pi}(k_BT)^S}(\frac{3}{2} - S)\Gamma(\frac{3}{2} - S) \qquad \text{(by expansion of Gamma function)}$$

Exercise (4) Boltzman transport eqn. In a uniform electric field

$$\frac{df}{dt} = -\frac{\partial f}{\partial v}\frac{d\vec{v}}{dt} - \frac{f - f_o}{\tau} = 0 \text{ in steady state}$$

$$\Rightarrow f - f_o = -q\frac{\vec{E}_o\tau}{m}\cdot\frac{\partial f}{\partial \vec{v}} = -\frac{q\tau\vec{E}_o}{m}\cdot\vec{\nabla}_v f$$

As in lecture (and McMelvey) we approximate $\vec{\nabla}_v f \cong \vec{\nabla}_v f_o$

So $$f \simeq f_o - \frac{q\tau\vec{E}_o}{m}\cdot\vec{\nabla}_v f_o \equiv f_o - \vec{v}_o\cdot\vec{\nabla}v_v f_o$$

once we define $\vec{v}_o \equiv q\tau\vec{E}_o/m$

Now recall form of Taylor's series for increment, h

$$P(x+h) = P(x) + hP'(x) + \ldots$$

By associating $x \to \vec{v}, h \to -\vec{v}_o, P \to f_o$

We find $f \cong f_o(\vec{v} - \vec{v}_o) = Ae^{-\frac{1}{2}m(\vec{v}-\vec{v}_o)^2/k_BT}$

The validity of this expression rests on accuracy of the Taylor expansion

$$f \simeq f_o - \frac{q\tau\vec{E}_o}{m}\cdot\vec{\nabla}f_o$$

$$\vec{\nabla}f_o = \frac{mv}{k_BT}f_o \Rightarrow f \simeq f_o - \frac{q\tau E_o}{m}\frac{mv}{k_BT}f_o$$

$$f = f_o(1 - v_o\frac{mv}{k_BT})$$

Validity of Taylor expansion requires that f is small deviation from

$$f_o \Rightarrow v_o\frac{mv}{kT} \ll 1$$

$$v_o\frac{mv}{k_BT} \approx \frac{mv_o^2}{k_BT} < 1 \Rightarrow v_o \ll \sqrt{\frac{k_BT}{m}}$$

Exercise (5) Conductivity for energy independent mean-free-path

Define mean-free-path, $\lambda = v\cdot\tau$; Apply Maxwell distribution

$$\sigma = \frac{ne^2 < \tau >}{m} \quad < \tau > = \frac{< \tau v^2 >}{< v^2 >} = \frac{\lambda < v >}{< v^2 >}$$

$$f_o = Ae^{-\frac{1}{2}mv^2 / k_B T}$$

$$< v > = \int_0^\infty Ave^{-\frac{1}{2}mv^2 / k_B T} v^2 dv = \int_0^\infty Av^4 e^{-\alpha v^2} dv$$

$$= \frac{4}{8}\frac{3}{}(\frac{k_B T}{m})^2 \sqrt{\frac{\pi k_B T}{\frac{1}{2}m}} A = \frac{3\sqrt{2\pi}}{2}(\frac{k_B T}{m})^{\frac{5}{2}} A = 3\sqrt{\frac{\pi}{2}}(\frac{k_B T}{m})^{\frac{5}{2}} \cdot A$$

So $$< \tau > = \frac{\lambda < v >}{< v^2 >} = \frac{\lambda 2A(k_B T)^2 / m^2}{\frac{3A\sqrt{2\pi}}{2}({k_B T}/{m})^{\frac{5}{2}}} = \lambda \frac{4}{3}\sqrt{\frac{m}{2\pi k_B T}}$$

$$< \sigma > = \frac{ne^2 < \tau >}{m} = \frac{4}{3}\frac{\lambda ne^2}{\sqrt{2\pi m k_B T}}$$

Alternatively, could do integrals using Γ function

Define $u = \frac{1}{2}mv^2 / kT$ $\qquad du = \frac{mv}{k_B T}dv$

$$du = \frac{m}{k_B T}\sqrt{\frac{2uk_B T}{m}}dv$$

$$du = \sqrt{\frac{2um}{k_B T}}dv$$

$$< v^2 > = \int_0^\infty A(\frac{2uk_B T}{m})^2 e^{-u}\sqrt{\frac{k_B T}{2um}}du$$

$$= \int_0^\infty A(\frac{k_B T}{m})^{\frac{5}{2}}(2u)^{\frac{3}{2}} e^{-u} du$$

$$\int_0^\infty u^{\frac{3}{2}} e^{-u} du = \Gamma(5/2) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{3}{2})$$
$$= \frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{3}{4}\Gamma(\frac{1}{2}) = \frac{3\sqrt{\pi}}{4}$$

So $< v^2 >= A(\frac{k_B T}{m})^{\frac{5}{2}} 2^{\frac{3}{2}} \cdot \frac{3\sqrt{\pi}}{4} = A(\frac{k_B T}{m})^{\frac{5}{2}} \cdot 3\sqrt{\frac{\pi}{2}}$
as derived with Gaussian integrals above