Homework #5(a) Solutions

Problem 1. Cyclotron resonance in spheroidal band

We start with the semiclassical dynamic equation, $\hbar \frac{d\vec{k}}{dt} = \frac{e}{\hbar} \frac{\partial U}{\partial \vec{k}} \times \vec{B}$. From the given spheroidal form for U(k), $\frac{\partial U}{\partial k_x} = \frac{\hbar^2 k_x}{m_T}, \frac{\partial U}{\partial k_y} = \frac{\hbar^2 k_y}{m_T}, \frac{\partial U}{\partial k_z} = \frac{\hbar^2 k_z}{m_L}$. So if we confine B to the x, y plane $\vec{B} = B_x \hat{x} + B_y \hat{y}$ the semi-classical equation becomes

$$\frac{dk_x}{dt} = \frac{-ek_z B_y}{m_L} \qquad \frac{dk_y}{dt} = \frac{ek_z B_x}{m_L} \qquad \frac{dk_z}{dt} = e\left(\frac{k_x B_y}{m_T} - \frac{k_y B_x}{m_T}\right) \tag{1}$$

Note: $$\frac{1}{\hbar^2}\frac{\partial U}{\partial \vec{k}} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{k_x}{m_T} & \frac{k_y}{m_T} & \frac{k_z}{m_L} \\ B_x & B_y & 0 \end{vmatrix} = \hat{x}\left(\frac{-k_z B_y}{m_L}\right) + \hat{y}\left(\frac{k_z B_x}{m_L}\right) + \hat{z}\left(\frac{k_x B_y}{m_T} - \frac{k_y B_x}{m_T}\right)$$

We seek oscillatory solutions in k space (would be oscillatory in real space too, but do not need that for the present problem). Hence,

$$\vec{k} = \vec{k}_0 e^{j\omega t}; \quad \frac{dk_x}{dt} = j\omega k_{0x}; \quad \frac{dk_y}{dt} = j\omega k_{0y}; \quad \frac{dk_z}{dt} = j\omega k_{0z}$$

and (1) becomes

$$0 = -j\omega k_{0x} - \frac{eB_y}{m_L} k_{0z} \qquad 0 = -j\omega k_{0y} + \frac{eB_x}{m_L} k_{0z}$$

$$0 = -j\omega k_{0z} + \frac{eB_y}{m_T} k_{0x} - \frac{eB_x}{m_T} k_{0y}$$

We can write this in elegant matrix form as

$$0 = \begin{pmatrix} -j\omega & 0 & \mp\frac{eB_y}{m_L} \\ 0 & -j\omega & \pm\frac{eB_x}{m_L} \\ \pm\frac{eB_y}{m_T} & \mp\frac{eB_x}{m_T} & -j\omega \end{pmatrix} \begin{bmatrix} k_{0x} \\ k_{oy} \\ k_{0z} \end{bmatrix}$$

From linear algebra we know this matrix equation has non-trivial solutions for the column vector $\vec{k}$ if and only if the matrix is singular, i.e., the determinant of the matrix vanishes. So

$$Det[\ ] = -j\omega\left(-\omega^2 + \frac{e^2 B_x^2}{m_T m_L}\right) \mp \frac{eB_y}{m_L}\left(\pm j\omega \frac{eB_y}{m_T}\right) = -j\omega\left(-\omega^2 + \frac{e^2 B_x^2}{m_T m_L} + \frac{e^2 B_y^2}{m_T m_L}\right) = 0$$

The non-trivial solution comes from inside the parenthesis.

$$\Rightarrow \omega^2 = \frac{e^2 B_x^2}{m_T m_L} + \frac{e^2 B_y^2}{m_T m_L} = \frac{e^2 |\vec{B}|^2}{m_T m_L}$$

or

$$\omega = \frac{e|\vec{B}|}{\sqrt{m_T m_L}}, |\vec{B}| = \sqrt{B_x^2 + B_y^2}$$

Problem 2. Hall Effect with two carrier types and spherical bands

Semi-classical equations:

Electrons: $$\hbar \frac{d\vec{k}_e}{dt} = -e\left(E + \frac{1}{\hbar}\nabla_k U_e(\vec{k}) \times \vec{B}\right) - \frac{\hbar \vec{k}_e}{\tau_e} = -e\left(\vec{E} + \hbar \frac{\vec{k}_e}{m_e^*} \times \vec{B}\right) - \frac{\hbar \vec{k}_e}{\tau_e}$$

Holes: $$\hbar \frac{d\vec{k}_h}{dt} = e\left(\vec{E} + \frac{1}{\hbar}\nabla_k U_h(\vec{k}) \times \vec{B}\right) - \frac{\hbar \vec{k}_h}{\tau_h} = e\left(\vec{E} + \frac{\hbar \vec{k}_h \times \vec{B}}{m_h^*}\right) - \frac{\hbar \vec{k}_h}{\tau_h}$$

In Hall configuration, $\vec{B} = B_0 \hat{z}$, $\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}, \Rightarrow$

$$\vec{k}_e \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_{ex} & k_{ey} & k_{ez} \\ 0 & 0 & B_0 \end{vmatrix} = \hat{x} k_{ey} B_0 - \hat{y} k_{ex} B_0; \quad \vec{k}_h \times \vec{B} = \hat{x} k_{hy} B_0 - \hat{y} k_{hx} B_0$$

In a steady state all time derivatives go to zero, so semi-classical equations become

Electrons

(1)
$$0 = -eE_x - \frac{e\hbar k_{ey} B_0}{m_e^*} - \frac{\hbar k_{ex}}{\tau_e}$$
$$0 = -eE_y + \frac{e\hbar k_{ex} B_0}{m_e^*} - \frac{\hbar k_{ey}}{\tau_e}$$
$$0 = -eE_z - \frac{\hbar k_{ez}}{\tau_e}$$

Holes

(2)
$$0 = eE_x + \frac{e\hbar k_{hy} B_0}{m_h^*} - \frac{\hbar k_{hx}}{\tau_h}$$
$$0 = eE_y - \frac{e\hbar k_{hx} B_0}{m_h^*} - \frac{\hbar k_{hy}}{\tau_h}$$
$$0 = eE_z - \frac{\hbar k_{hz}}{\tau_h}$$

These is just six equations in six unknowns, $k_{ex}, k_{ey}, k_{ez}, k_{hx}, k_{hy}, k_{hz}$. To solve, multiply the first of set (1) by $eB_0\tau_e / m_e^*$ and add to the second of set (1). Then multiply the first of set (2) by $-eB_0\tau_h / m_h^*$ and add to the second of set (2). We get

$$0 = \frac{-e^2 B_0 \tau_e E_x}{m_e^*} - \frac{e^2 \hbar k_{ey}}{(m_e^*)^2} - eE_y - \frac{\hbar k_{ey}}{\tau_e}$$

$$0 = \frac{-e^2 E_x B_0 \tau_h}{m_h^*} - \frac{-e^2 \hbar k_{hy} B_0^2 \tau_h}{(m_h^*)^2} + eE_y - \frac{\hbar k_{hy}}{\tau_h}$$

These can be rewritten as:

$$k_{ey}\left[\frac{-e^2\hbar B_0^2\tau_e}{\left(m_e^*\right)^2}-\frac{\hbar}{\tau_e}\right]=eE_y+\frac{e^2B_0\tau_eE_x}{m_e^*}$$ and $$k_{hy}\left[\frac{-e^2B_0^2\tau_h}{\left(m_h^*\right)^2}-\frac{\hbar}{\tau_h}\right]=-eE_y+\frac{e^2E_xB_0\tau_h}{m_h^*}$$

Ignoring terms in B_0^2, we get

$$k_{ey}\approx\frac{-\tau_e}{\hbar}\left(eE_y+\frac{e^2B_0\tau_eE_x}{m_e^*}\right) \qquad k_{hy}\approx\frac{\tau_h}{\hbar}\left(eE_y-\frac{e^2B_0\tau_hE_x}{m_h^*}\right)$$

Recall that the electron current in a given band is defined in the semiclassical model as

$$J_{ey}=-eE_y\int_{band}\frac{\hbar k_{ey}}{m_e^*}f_ed\vec{k}_e=\frac{e\hbar}{m_e^*}\frac{\tau_e}{\hbar}\left(eE_y+\frac{e^2B_0\tau_eE_x}{m_e^*}\right)\underbrace{\int f_ed^3k}_{n}$$

Similarly, the hole electrical current is

$$J_{hy}=eE_y\int_{band}\frac{\hbar k_h}{m_h^*}f_hd\vec{k}_h=\frac{e\hbar}{m_h^*}\frac{\tau_h}{\hbar}\left(eE_y-\frac{e^2B_0\tau_hE_x}{m_h^*}\right)\underbrace{\int f_hd^3k}_{p}$$

Since the net current along y axis is zero, we can write $J_{ey}+J_{hy}=0$ or

$$0=\frac{e^2\tau_e}{m_e^*}\left(E_y+\frac{eB_0\tau_eE_x}{m_e^*}\right)n+\frac{e^2\tau_h}{m_h}\left(E_y-\frac{eB_0\tau_hE_x}{m_h^*}\right)p$$

Solving for E_y we get $$\left(ne\mu_e+pe\mu_h\right)E_y=\left[\frac{ne^3\tau_e}{m_e^*}\left(\frac{-B_0\tau_e}{m_e^*}\right)+\frac{pe^3\tau_h}{m_h^*}\left(\frac{B_0\tau_h}{m_h^*}\right)\right]E_x$$

where $\mu_e=\frac{e\tau_e}{m_e^*},\quad \mu_h=\frac{e\tau_h}{m_h^*}$ as in kinetic theory. Hence, we can write

$$\left(n\mu_e+p\mu_h\right)E_y=B_0E_x\left(-n\mu_e^2+p\mu_h^2\right)$$

or $$\frac{E_y}{E_xB_0}=\frac{-n\mu_e^2+p\mu_h^2}{n\mu_e+p\mu_h}$$

and $$R_H \equiv \frac{E_y}{J_x B_0} = \frac{E_y}{e(\sigma_e + \sigma_h) E_x B_0} \quad \text{where} \quad \sigma_e = ne\mu_e, \quad \sigma_h = pe\mu_h$$

Finally, $$R_H = \frac{-n\mu_e^2 + p\mu_h^2}{e(n\mu_e + p\mu_h)^2} = \frac{\mu_h^2\left(p - n\frac{\mu_e^2}{\mu_h^2}\right)}{e\mu_h^2\left(p + n\frac{\mu_e}{\mu_h}\right)^2} = \frac{p - nb^2}{e(p + nb)}, \quad \text{where} \quad b \equiv \frac{\mu_e}{\mu_h}$$

Important point: This result is valid for any orientation of $\vec{B}$ in x-y plane and independent of the particle charge polarity. But the given form of spheroidal constant energy surface is only precise for the electrons in the conduction band of indirect band-gap semiconductors, such as Si and Ge.