6. Energy dependent scattering time of generic form: \( \tau(U) = AU^{-S} \). From the text

\[
<\tau> = \frac{m}{3k_B T} \int \tau(v)f_o(v)v^2 dv = \frac{2}{3k_B T} \int U^{3/2}\tau(U)f_o(U)dU
\]

where \( f_o(u) = n\left(\frac{h^2}{2\pi mk_B T}\right)^{3/2}e^{-U/k_BT} \) (low-density particle distribution function)

So

\[
<\tau> = \frac{2}{3k_B T} \int_0^\infty U^{3/2}Au^{-s}e^{-U/k_BT}dU
\]

\[
U' = \frac{U}{k_BT} \Rightarrow dU = k_BTdU'
\]

\[
<\tau> = \frac{2}{3k_B T} \frac{(k_BT)^{3/2}(k_BT)^{-s}\int_0^\infty AU^{3/2-S}e^{-U'}dU'}{(k_BT)^{3/2}\int_0^\infty (U')^{1/2}e^{-U'}dU'}
\]

Now recall definition \( \Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \). Thus by expansion of the Gamma function

\[
\int_0^\infty (U')^{1/2}e^{-U'}dU' = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}
\]

\[
\int_0^\infty (U')^{3/2-S}e^{-U'}dU' = \Gamma\left(\frac{5}{2} - S\right) = \left(\frac{3}{2} - S\right)\Gamma\left(\frac{3}{2} - S\right)
\]

So

\[
<\tau> = \frac{2A}{3k_BT}(k_BT)^{-S} \frac{\Gamma\left(\frac{5}{2} - S\right)}{\sqrt{\pi}^{3/2}} = \frac{4A}{3\sqrt{\pi}(k_BT)^s} \Gamma\left(\frac{5}{2} - S\right) \\
= \frac{4A}{3\sqrt{\pi}(k_BT)^s} \left(\frac{3}{2} - S\right)\Gamma\left(\frac{3}{2} - S\right)
\]
Note: the gamma function is not as mysterious as it might first appear. In fact, it included in Microsoft Excel as a worksheet function gammaln(x), that is mathematically equal to ln[\( \Gamma(x) \)].

From applied mathematics, we know that \( \Gamma(n) \) has a special definition when \( n \) is an integer in terms of the factorial function, \( \Gamma(n) = (n-1)! \). This is seen in the Figure, for example at \( S = -2.5 \), \( \Gamma[5/2-(-5/2)] = \Gamma(5) = (5-1)! = 24 \).

7. Boltzmann transport in a uniform electric field

\[
\frac{df}{dt} = -\frac{\partial f}{\partial \vec{v}} \frac{d\vec{v}}{dt} - \frac{f - f_o}{\tau} = 0 \text{ in the steady state}
\]

\[\Rightarrow f - f_o = -q \frac{\vec{E}_o \tau}{m} \cdot \frac{\partial f}{\partial \vec{v}} = -\frac{q \tau \vec{E}_o}{m} \cdot \vec{v} \cdot f\]

As in the notes we approximate \( \vec{v}_v f \cong \vec{v}_{v} f_o \)

So \( f \cong f_o - \frac{q \tau \vec{E}_o}{m} \cdot \vec{v}_v f_o \equiv f_o - \vec{v}_o \cdot \vec{v}_v f_o \)

Once we define \( \vec{v}_o \equiv q \tau \vec{E}_o / m \)

Now recall form of Taylor’s series for increment, \( h \)

\[
P(x + h) = P(x) + hP'(x) + ...
\]

By associating \( x \rightarrow \vec{v}, h \rightarrow -\vec{v}_o, P \rightarrow f_o \).
We find

\[ f \equiv f_o(\vec{v} - \vec{v}_o) = Ae \frac{1}{2} m(\vec{v} - \vec{v}_o)^2 / k_B T \]

The validity of this expression rests on accuracy of the Taylor expansion

\[ f = f_o - \frac{q\tau E_o}{m} \cdot \vec{v} f_o \]

\[ \vec{v} f_o = \frac{mv}{k_B T} f_o \Rightarrow f = f_o - \frac{q\tau E_o \cdot \frac{mv}{k_B T} f_o}{m} \]

\[ f = f_o \left( 1 - \frac{mv}{k_B T} \right) \]

Validity of Taylor expansion requires that \( f \) is small deviation from \( f_0 \), which implies

\[ f_o \Rightarrow \frac{mv}{k_B T} \ll 1 \]

or

\[ \frac{mv}{k_B T} \approx \frac{mv^2}{k_B T} < 1 \Rightarrow v_o \ll \sqrt{\frac{k_B T}{m}} \]

8. Electrical conductivity for energy-independent mean-free-path

Define mean-free-path, \( \lambda = v \cdot \tau \); Apply Maxwell distribution

\[ \sigma = \frac{ne^2 < \tau >}{m} ; \quad < \tau > = \frac{< \tau v^2 >}{< v^2 >} = \frac{\lambda < v >}{< v^2 >} \]

\[ f_o = Ae^{-(1/2)mv^2 / k_B T} \]

\[ < v > = \int_0^\infty Av e^{-(1/2)mv^2 / k_B T} v^2 dv = \int_0^\infty Av^3 e^{-\lambda v^2} dv \]

\[ < v > = \frac{A}{2\alpha^2} = \frac{2A(k_B T)^2}{m^2} \]

\[ = \frac{4}{8} 3(k_B T / m)^2 \sqrt{\frac{\pi k_B T}{m / 2}} A = \frac{3\sqrt{2\pi}}{2} (k_B T / m)^{5/2} A = 3\sqrt{\frac{\pi}{2}} (k_B T / m)^{5/2} \cdot A \]

So

\[ < \tau > = \frac{\lambda < v >}{< v^2 >} = \frac{2\lambda A(k_B T)^2 / m^2}{(3A\sqrt{2\pi / 2})(k_B T / m)^{5/2}} = \lambda \frac{4\sqrt{m}}{3\sqrt{2\pi k_B T}} \]
\[
< \sigma > = \frac{ne^2}{m} \frac{\langle \tau \rangle}{3\sqrt{2\pi m k_B T}}
\]

Alternatively, could do integrals using \( \Gamma \) function

Define

\[
U = \frac{1}{2} \frac{mv^2}{k_B T} \quad dU = \frac{mv}{k_B T} dv
\]

\[
dU = \frac{m}{k_B T} \sqrt{\frac{2Uk_B T}{m}} dv
\]

\[
dU = \frac{2mU}{k_B T} dv
\]

\[
< v^2 > = \int_0^\infty A(2Uk_B T / m)^{5/2} e^{-U} \frac{k_B T}{\sqrt{2mU}} dU
\]

\[
= \int_0^\infty A(k_B T / m)^{5/2} (2U)^{3/2} e^{-U} dU
\]

\[
\int_0^\infty U^{3/2} e^{-U} dU = \Gamma(5/2) = \Gamma(3/2 + 1) = 3/2 \cdot \Gamma(3/2)
\]

\[
= (3/2)\Gamma(1/2 + 1) = 3/4 \Gamma(1/2) = 3\sqrt{\pi} / 4
\]

So

\[
< v^2 > = A(k_B T / m)^{5/2} 2^{3/2} \cdot 3\sqrt{\pi} / 4 = A(k_B T / m)^{5/2} \cdot 3\sqrt{\pi} / 2
\]

as derived with Gaussian integrals above.