Problem 1. Semiclassical correction to kinetic theory
(a) N-type phosphorous doped Si \(1 \times 10^{18}/\text{cm}^3\), \(\delta = 0.02\Omega - \text{cm}\)

We know \(\sigma = ne\mu\) so, \(\rho \equiv 1/\sigma = 1/(ne\mu) \Rightarrow \mu = 1/(ne\rho)\). In practical units,
\[\mu = \left(10^{18}\right)\left(1.6e^{-19}\right)(0.02) = 312 \text{ cm}^2/(\text{V-s})\] or in MKS units \(3.12\times10^{-2} \text{ m}^2/(\text{V-s})\)

(b) \(\mu = e\tau / m^* \Rightarrow \tau = m^*\mu / e\). The correct effective mass to use for the scattering time is
the conductivity mass \(m_e / m_i^* = (1/6)(2/0.98 + 4/0.19) \Rightarrow m_i^* = 0.26m_i\) since scattering involves dynamic mechanical effects. Substitution of MKSA values yields \(\tau = 46\text{ fs}\).

(c) Einstein relation \(D = k_B T \mu / e = 8.06\times10^{-4} \text{ m}^2/\text{s}\).

Problem 2. Cyclotron resonance in spheroidal band

We start with the semiclassical dynamic equation,
\[\hbar \frac{dk}{dt} = \frac{e}{\hbar} \frac{\partial U}{\partial k} \times \vec{B} \]. From the given

spheroidal form for \(U(k)\),
\[\frac{\partial U}{\partial k_x} = \hbar^2 k_x / m_T, \quad \frac{\partial U}{\partial k_y} = \hbar^2 k_y / m_T, \quad \frac{\partial U}{\partial k_z} = \hbar^2 k_z / m_i\]. So if we confine \(B\) to the

\(x, y\) plane \(\vec{B} = B_x \hat{x} + B_y \hat{y}\) the semi-classical equation becomes

\[\begin{align*}
\frac{dk_x}{dt} &= -\frac{ek_z B_x}{m_L} \\
\frac{dk_y}{dt} &= \frac{ek_z B_x}{m_L} \\
\frac{dk_z}{dt} &= e \left( \frac{k_x B_y}{m_T} - \frac{k_y B_x}{m_T} \right)
\end{align*}\] (1)

Note:
\[\frac{1}{\hbar^2} \frac{\partial U}{\partial k} \times \vec{B} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
k_x & k_y & k_z \\
B_x & B_y & 0
\end{vmatrix} = \hat{x} \left( -\frac{k_x B_y}{m_L} \right) + \hat{y} \left( \frac{k_x B_z}{m_L} \right) + \hat{z} \left( \frac{k_y B_x}{m_T} - \frac{k_y B_x}{m_T} \right)\]

We seek oscillatory solutions in \(k\) space (would be oscillatory in real space too, but do not

need that for the present problem). Hence,
\[\vec{k} = \vec{k}_0 e^{j\omega t}; \quad \frac{dk_x}{dt} = j\omega k_{0x}; \quad \frac{dk_y}{dt} = j\omega k_{0y}; \quad \frac{dk_z}{dt} = j\omega k_{0z}\]

and (1) becomes
We can write this in elegant matrix form as

\[
0 = \begin{pmatrix}
-j\omega & 0 & \frac{\pm eB_y}{m_L} \\
0 & -j\omega & \frac{\pm eB_x}{m_L} \\
\frac{\pm eB_y}{m_T} & \frac{\pm eB_x}{m_T} & -j\omega
\end{pmatrix}
\begin{pmatrix}
k_{0x} \\
k_{0y} \\
k_{0z}
\end{pmatrix}
\]

From linear algebra we know this matrix equation has non-trivial solutions for the column vector \( \vec{k} \) if and only if the matrix is singular, i.e., the determinant vanishes. So

\[
\text{Det} \left[ \begin{array}{ccc}
-j\omega & 0 & \frac{\pm eB_y}{m_L} \\
0 & -j\omega & \frac{\pm eB_x}{m_L} \\
\frac{\pm eB_y}{m_T} & \frac{\pm eB_x}{m_T} & -j\omega
\end{array} \right] = 0
\]

The non-trivial solution comes from inside the parenthesis.

\[
\Rightarrow \omega^2 = \frac{e^2 B_x^2}{m_T m_L} + \frac{e^2 B_y^2}{m_T m_L} = \frac{e^2 |\vec{B}|^2}{m_T m_L}
\]

or

\[
\omega = \frac{e |\vec{B}|}{\sqrt{m_T m_L}}, |\vec{B}| = \sqrt{B_x^2 + B_y^2}
\]

**Problem 3. Semiclassical transport in germanium**

(a) For any energy band \( \vec{v}_g = (\hbar^{-1}) \vec{V} \left( \vec{k} \right) = v_{g_x} \hat{x} + v_{g_y} \hat{y} + v_{g_z} \hat{z} \) and \( d\vec{r} / dt = \vec{v}_g \), where \( \vec{r} \) defines the location of the wave-packet center. For the specific energy band

\[
U = (\hbar^2 / 2)(\alpha k_x^2 + \alpha k_y^2 + \alpha k_z^2 + 2\beta k_x k_y + 2\beta k_x k_z + 2\beta k_y k_z),
\]

we get

\[
\vec{v}_{g_x} = \hbar(\alpha k_x + \beta(k_x + k_z)); \quad \vec{v}_{g_y} = \hbar(\alpha k_y + \beta(k_y + k_z)); \quad \vec{v}_{g_z} = \hbar(\alpha k_z + \beta(k_x + k_y)),
\]

and thus \( dx / dt = \hbar(\alpha k_x + \beta(k_y + k_z));\)

\( dy / dt = \hbar(\alpha k_y + \beta(k_x + k_z));\)

\( dz / dt = \hbar(\alpha k_z + \beta(k_x + k_y)),\)
(b) The vector semiclassical equation of motion is \( \hbar \vec{k} / dt = q \left( \vec{E} + e^{-i \hat{\nabla}_n \vec{U} \times \vec{B}} \right) - \hbar \vec{k} / \tau \).

For \( \vec{E} = E_0 \hat{x} \) and \( \vec{B} = B_0 \hat{z} \), the component equations become

\[
x : \frac{h \dot{k}_x}{dt} = q \left[ E_0 + B_0 [\alpha k_y + \beta (k_x + k_z)] \right] - \hbar k_x / \tau
\]

\[
y : \frac{h \dot{k}_y}{dt} = B_0 [\alpha k_x + \beta (k_y + k_z)] - \hbar k_y / \tau \quad \text{and} \quad z : \frac{h \dot{k}_z}{dt} = -\hbar k_z / \tau.
\]

In the steady state, \( \dot{k}_x / dt = 0 \), so if \( B_0 = 0 \), the x component has obvious solution \( k_x = qE_0 \tau / \hbar \).

(c) From (a), the motion of the wavepacket along x is \( dx / dt = \hbar (\alpha k_x + \beta (k_y + k_z)) \).

Assuming \( B = 0 \), the only nonzero ballistic terms from (b) are for \( k_x \) (ballistic means collisionless, so \( \tau \) is set to infinity): \( k_x = qE_0 \tau / \hbar + C \) where \( C \) is a constant that must be zero if particle starts at rest (rest means zero velocity and hence \( k_x = 0 \)). Substitution yields \( dx / dt = \hbar \alpha (qE_0 \tau / \hbar) \) or \( x = \hbar \alpha (qE_0 \tau^2 / 2 \hbar) + D \), where \( D \) is a second constant. The distance traveled in real space between \( t = 0 \) and \( t = \tau \) is just \( \Delta x = x(t = \tau) - x(t = 0) = \hbar \alpha (qE_0 \tau^2 / 2 \hbar) \)

**Problem 4. Hall Effect with two carrier types and spherical bands**

Semi-classical equations:

Electrons:
\[
\hbar \frac{d\vec{k}_e}{dt} = -e \left( E + \frac{1}{\hbar} \hat{\nabla}_n \vec{U}_e (\vec{k}) \times \vec{B} \right) - \hbar \frac{\vec{k}_e}{\tau_e} = -e \left( \vec{E} + \frac{\hbar \vec{k}_e}{m_e} \times \vec{B} \right) - \frac{\hbar \vec{k}_e}{\tau_e}
\]

Holes:
\[
\hbar \frac{d\vec{k}_h}{dt} = e \left( \vec{E} + \frac{1}{\hbar} \hat{\nabla}_n \vec{U}_h (\vec{k}) \times \vec{B} \right) - \hbar \frac{\vec{k}_h}{\tau_h} = e \left( \vec{E} + \frac{\hbar \vec{k}_h}{m_h} \times \vec{B} \right) - \frac{\hbar \vec{k}_h}{\tau_h}
\]

In Hall configuration, \( \vec{B} = B_0 \hat{z} \), \( \vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \), \( \Rightarrow \)

\[
\vec{k}_e \times \vec{B} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
k_{ex} & k_{ey} & k_{ez} \\
0 & 0 & B_0
\end{vmatrix} = \hat{x} k_{ex} B_0 - \hat{y} k_{ey} B_0 \quad \text{and} \quad \vec{k}_h \times \vec{B} = \hat{x} k_{hx} B_0 - \hat{y} k_{hy} B_0
\]

In a steady state all time derivatives go to zero, so semi-classical equations become:
Electrons

\[ 0 = -eE_x - \frac{\epsilon h k_{ey} B_0}{m_e^*} - \frac{\hbar k_{ey}}{\tau_e} \]

\[ 0 = -eE_y + \frac{\epsilon h k_{ex} B_0}{m_e^*} - \frac{\hbar k_{ex}}{\tau_e} \]

\[ 0 = -eE_z - \frac{\hbar k_{ez}}{\tau_e} \]

(1)

Holes

\[ 0 = eE_x + \frac{\epsilon h k_{hx} B_0}{m_h^*} - \frac{\hbar k_{hx}}{\tau_h} \]

\[ 0 = eE_y - \frac{\epsilon h k_{hy} B_0}{m_h^*} - \frac{\hbar k_{hy}}{\tau_h} \]

\[ 0 = eE_z - \frac{\hbar k_{hz}}{\tau_h} \]

(2)

These is just six equations in six unknowns, \( k_{ex}, k_{ey}, k_{ez}, k_{hx}, k_{hy}, k_{hz} \). To solve, multiply the first of set (1) by \( eB_0 \tau_e / m_e^* \) and add to the second of set (1). Then multiply the first of set (2) by \(-eB_0 \tau_h / m_h^*\) and add to the second of set (2). We get

\[ 0 = -\frac{e^2 B_0 \tau_e E_x}{m_e^*} - \frac{e^2 \hbar k_{ey}}{(m_e^*)^2} - eE_y - \frac{\hbar k_{ey}}{\tau_e} \]

\[ 0 = -\frac{e^2 E_x B_0 \tau_h}{m_h^*} - \frac{e^2 \hbar k_{hy} B_0^2 \tau_h}{(m_h^*)^2} + eE_y - \frac{\hbar k_{hy}}{\tau_h} \]

These can be rewritten as:

\[ k_{ey} \left[ -\frac{e^2 \hbar B_0^2 \tau_e}{(m_e^*)^2} - \frac{\hbar}{\tau_e} \right] = eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*} \quad \text{and} \quad k_{hy} \left[ -\frac{e^2 B_0 \tau_h}{(m_h^*)^2} - \frac{\hbar}{\tau_h} \right] = -eE_y + \frac{e^2 E_x B_0 \tau_h}{m_h^*} \]

Ignoring terms in \( B_0^2 \), we get

\[ k_{ey} \approx -\frac{\tau_e}{\hbar} \left( eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*} \right) \quad \text{and} \quad k_{hy} \approx \frac{\tau_h}{\hbar} \left( eE_y - \frac{e^2 B_0 \tau_e E_x}{m_e^*} \right) \]

Recall that the electron current in a given band is defined in the semiclassical model as

\[ J_{ey} = -eE_y \int_{\text{band}} \frac{\hbar k_{ey}}{m_e^*} f_e \, d\vec{k} = e \frac{\tau_e}{\hbar} \int_{\text{band}} \left( eE_y + \frac{e^2 B_0 \tau_e E_x}{m_e^*} \right) f_e \, d^3k \]
Similarly, the hole electrical current is

\[ J_{hy} = eE_y \int_{band} \frac{\hbar k}{m_h} f_h \, d\vec{k} = \frac{e\hbar}{m_h} \left( eE_y - \frac{e^2 B_0 \tau_h E_x}{m_h} \right) \int f_h \, d^3k \]

Since the net current along y axis is zero, we can write \( J_{ey} + J_{hy} = 0 \) or

\[ 0 = \frac{e^2 \tau_e}{m_e} \left( E_y + \frac{eB_0 \tau_e E_x}{m_e} \right) n + \frac{e^2 \tau_h}{m_h} \left( E_y - \frac{eB_0 \tau_h E_x}{m_h} \right) p \]

\[ \left( ne\mu_e + pe\mu_h \right) E_y = \left[ \frac{ne^3 \tau_e}{m_e} \left( -B_0 \frac{\tau_e}{m_e} \right) + pe^3 \tau_h \left( B_0 \frac{\tau_h}{m_h} \right) \right] E_x \]

Solving for \( E_y \) we get

\[ \mu_e = \frac{e\tau_e}{m_e}, \quad \mu_h = \frac{e\tau_h}{m_h} \]

where \( \mu_e, \mu_h \) as in kinetic theory. Hence, we can write

\[ \left( n\mu_e + p\mu_h \right) E_y = B_0 E_x \left( -n\mu_e^2 + p\mu_h^2 \right) \]

\[ \frac{E_y}{E_x B_0} = \frac{-n\mu_e^2 + p\mu_h^2}{n\mu_e + p\mu_h} \]

or

\[ R_{ji} \equiv \frac{E_y}{J_x B_0} = \frac{E_y}{e(\sigma_e + \sigma_h) E_x B_0} \]

where \( \sigma_e = ne\mu_e, \quad \sigma_h = pe\mu_h \)

and

\[ R_H = \frac{-n\mu_e^2 + p\mu_h^2}{e(n\mu_e + p\mu_h)^2} = \frac{\mu_e^2 \left( p - n\frac{\mu_e^2}{\mu_h^2} \right)}{e \mu_h^2 \left( p + n\frac{\mu_e^2}{\mu_h^2} \right)} = \frac{p - nb^2}{e(p + nb)}, \quad \text{where} \quad \frac{b}{\mu_h} = \frac{\mu_e}{\mu_h} \]

Important point: This result is valid for any orientation of \( \vec{B} \) in x-y plane and independent of the particle charge polarity. But the given form of spheroidal constant energy surface is only precise for the electrons in the conduction band of indirect band-gap semiconductors, such as Si and Ge.