6.1:

a)

$$\Delta\beta = \frac{k_0^2}{2\beta}\frac{\int \Delta\epsilon\,|U|^2\,dA}{\int |U|^2\,dA} \tag{6.9}$$

$k_0 = 2\pi/\lambda_0$

$\beta = 2\pi\bar{n}/\lambda_0$

$\Delta\epsilon = 2n\Delta n$ where n is the index of the guiding region.

$$\Delta\beta = \left(\frac{2\pi}{\lambda_0}\right)\frac{n\Delta n}{\bar{n}}\frac{\int\int_0^{150\text{ nm}} |U|^2\,dA}{\int\int_{-\infty}^{\infty} |U|^2\,dA}$$

Assuming that the mode doesn't change significantly due to the index perturbation, the fraction containing the integrals is the confinement factor of the original mode ($\Gamma = 0.5$ given by problem) multiplied by the overlap of the part of the guide that changed ($\frac{150\text{ nm}}{300\text{ nm}}$).

$$\Delta\beta = \frac{2\pi}{\lambda_0}\frac{n}{\bar{n}}\Delta n\Gamma_{xy} = \frac{2\pi}{1.55\mu\text{m}}(0.02)\frac{1}{4} = 203\text{ cm}^{-1} \tag{6.12, 6.10}$$

(Note that this assumes that $n \approx \bar{n}$ in the waveguide region.)

b) Express the $\Delta\epsilon$ ($\equiv 2n\Delta n$) term as

$$\Delta\epsilon = \pm n\Delta n$$

This makes the zeroth order Fourier term of $\Delta\epsilon$ equal to zero.

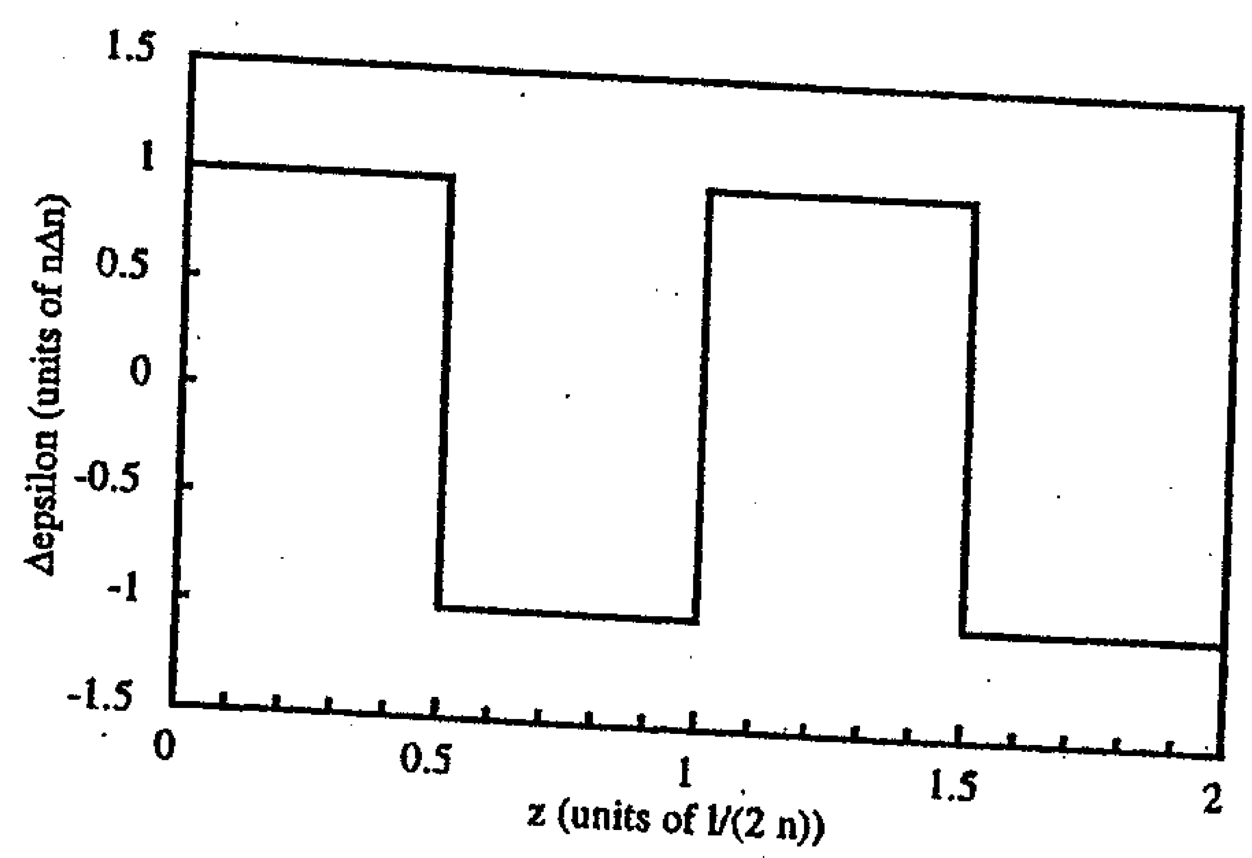

Figure 6.1b. Index perturbation profile.

Then the Fourier expansion of $\Delta\epsilon$ can be calculated by a standard method:

$$\Delta\epsilon(z) = \sum_l \Delta\epsilon_l\, e^{-j2\pi(z/\Lambda)l} \tag{6.16}$$

$$\int_0^{\Lambda} \Delta\epsilon(z)\; e^{j2\pi(z/\Lambda)m)}\,dz = \int_0^{\Lambda}\left(\sum_l \Delta\epsilon_l\, e^{j2\pi(z/\Lambda)(-l+m)}\right)\,dz$$

Simplify the right side using the orthogonality of exponentials:

$$\int_0^{\Lambda/2} n\Delta n\; e^{j2\pi(z/\Lambda)m)}\,dz \int_{\Lambda/2}^{\Lambda} -n\Delta n\; e^{j2\pi(z/\Lambda)m)}\,dz = \int_0^{\Lambda}(\Delta\epsilon_m)\,dz$$

$$4\left(\frac{-\Lambda\Delta nn}{j2\pi m}\right) = \Lambda\epsilon_m$$
$$0 = \Lambda\epsilon_m$$

$$\Delta\epsilon_l = \begin{cases} n\Delta n\frac{2j}{l\pi} & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases}$$

$$\Delta\epsilon(z) = n\Delta n \sum_{l\ odd} \left(\frac{2j}{l\pi}\right) e^{-j2\pi(z/\Lambda)l}$$

Then calculate κ:

$$\begin{aligned}\kappa_l &= \frac{k_0^2}{2\beta}\frac{\int\int \Delta\epsilon_l(x,y)|U|^2 dA}{\int\int |U|^2 dA} \\ &= \frac{\pi}{\lambda_0\bar{n}}\Delta\epsilon_l\Gamma_{xy_g} \\ &= \frac{\pi}{\lambda_0\bar{n}}\left(\frac{2j}{l\pi}n\Delta n\right)\Gamma_{xy_g} \\ &= j\frac{2\Delta n}{l\lambda_0}\Gamma_{xy_g}\end{aligned}$$

(Note that this assumes that $n \approx \bar{n}$ in the waveguide region.)

$$\begin{aligned}\kappa &= |\kappa_{l=1}| \\ &= \left|j\frac{2(0.02)}{1.55\mu\text{m}}\frac{1}{4}\right| \\ &= 64.5\ \text{cm}^{-1}\end{aligned}$$

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c) For the lossless case, the first order grating gives a reflection of

$$r_g = -j\frac{\tilde{\kappa}_{-1}\tanh(\sigma L)}{\sigma + j\delta\tanh(\sigma L)}$$

$\delta = \beta - \beta_0 = \frac{2\pi n}{\lambda_0} - \frac{\pi}{\Lambda}$

$\sigma = \sqrt{\kappa^2 - \delta^2}$

For the first order grating, we have from part (b) that

$$\kappa = |\kappa_{-1}| = 64.5\ \text{cm}^{-1}$$

At the Bragg wavelength ($\lambda_0 = 1.55\mu$m),

$$\delta = 0$$
$$\sigma = \kappa$$

$$|r_g| = \frac{\kappa\tanh(\kappa L)}{\kappa}$$

$$|r_g| = \sqrt{|R_g|} = \sqrt{0.5}$$

Solve for L:

$$L = 137\mu\text{m}$$

Solutions by R. Kehl Sink