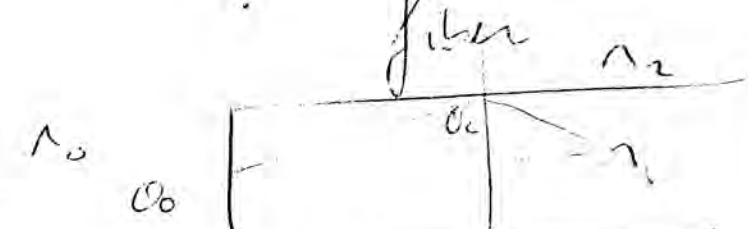
Step index optical Fibers

Now we can apply some of what we learned for the slab waveguide to the fiber waveguide

The step index fiber is the simplest optical fiber in terms of analysis.

Define: $\Delta = \dfrac{n_1^2 - n_2^2}{2n_1^2} \approx \dfrac{n_1 - n_2}{n_1} = \dfrac{\delta n}{n_1}$ for $n_1 \approx n_2$

Define: $NA = n_0 \sin\theta_0 = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{2\Delta}$
where θ_0 is the maximum input angle for $\theta > \theta_c$ inside fiber



$\boxed{\text{HW problem}}$

Cylindrical Waveguides

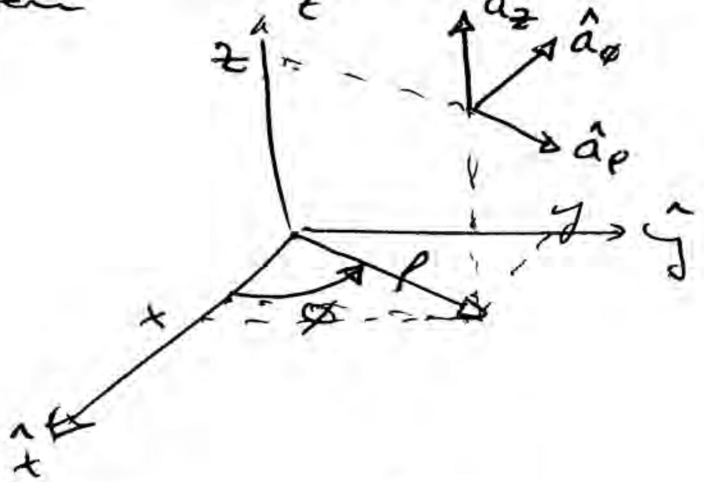
Summary:
- Standing waves are created in the "radial" directions
- Periodic waves are created in the ϕ direction
- Traveling waves are created in the z-direction
- Field must decay at $\rho = \infty$ and be finite at $\rho = 0$

General Solution is of the form

$$E(\rho, \phi, z) = E_0 J_m(\beta_\rho \rho) \cdot \left[C_2 \cos(m\phi) + D_2 \sin(m\phi) \right] \cdot \left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z} \right]$$

Cylindrical Coordinate System

Consider a system that is better represented in cylindrical coordinate system



A general solution for source-free, lossless media

$$E(\rho, \phi, z) = E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z$$

Substituting into the W.E.

$$\nabla^2 (E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z) + \beta^2 (E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z) = 0$$

Since $\nabla^2 (E_z \hat{a}_z) = a_z \nabla^2 E_z$

we can write

$$\nabla^2 E_z + \beta^2 E_z = 0$$

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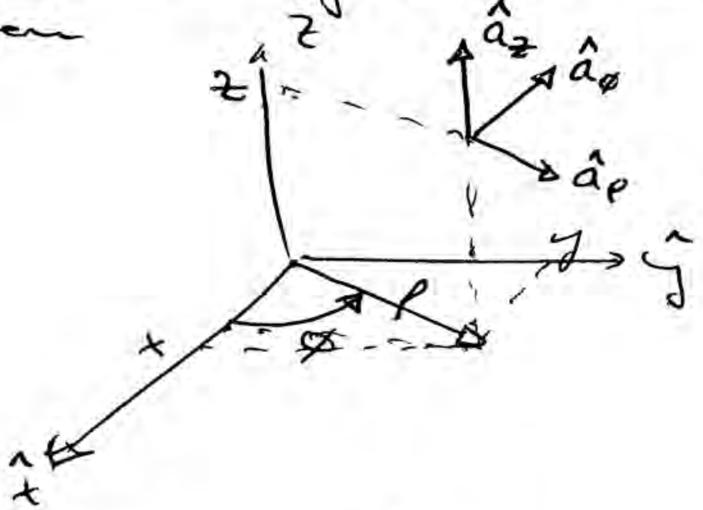

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we can write

$$\nabla^2 E_z + \beta^2 E_z = 0$$

Writing $\nabla^2 \bar{E} + \beta^2 \bar{E} = 0$

$$\nabla(\nabla \cdot \bar{E}) - \nabla \times \nabla \times \bar{E} + \beta^2 \bar{E} = 0$$

$$\nabla\left(\nabla \cdot \left[E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z\right]\right)$$
$$- \nabla \times \nabla \times \left[E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z\right]$$
$$+ \beta^2 \left[E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z\right] = 0$$

using $\nabla E = \frac{\partial E}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial E}{\partial \phi} \hat{a}_\phi + \frac{\partial E}{\partial z} \hat{a}_z$

rect. $\nabla \times \bar{E} = \hat{a}_x \left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \hat{a}_y \left[\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right] + \hat{a}_z \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$

cylind. $\nabla \times \bar{E} = \hat{a}_\rho \left[\frac{1}{\rho} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\phi}{\partial z}\right] + \hat{a}_\phi \left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right] + \hat{a}_z \left[\frac{1}{\rho} \frac{\partial \rho E_\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial E_\rho}{\partial \phi}\right]$

we can write

$$\nabla^2 E_\rho + \nabla^2 E_\phi + \nabla^2 E_z$$

$$- \nabla \times \left[\hat{a}_\rho \left[\frac{1}{\rho} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\phi}{\partial z}\right] + \hat{a}_\phi \left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right] + \hat{a}_z \left[\frac{1}{\rho} \frac{\partial \rho E_\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial E_\rho}{\partial \phi}\right]\right]$$
$$+ \beta^2 \left[E_\rho \hat{a}_\rho + E_\phi \hat{a}_\phi + E_z \hat{a}_z\right] = 0$$

⋮

One more step

$$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial \phi}\right) + \beta^2 E_\rho = 0$$

$$\nabla^2 E_\phi + \left(-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi}\right) + \beta^2 E_\phi = 0$$

$$\nabla^2 E_z + \beta^2 E_z = 0$$

where $\nabla^2 E(\rho, \phi, z) = \frac{\partial^2 E}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial \phi^2} + \frac{\partial^2 E}{\partial z^2}$

Expanding

$$\nabla^2 E_z + \beta^2 E_z = 0$$

$$\frac{\partial^2 E}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial \phi^2} + \frac{\partial^2 E}{\partial z^2} + \beta^2 E = 0$$

Assuming E is separable in ρ, ϕ & z

i.e. $E(\rho, \phi, z) = f(\rho) g(\phi) h(z)$

we can write (dividing through by $f \cdot g \cdot h$)

$$\frac{1}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f\rho}\frac{df}{d\rho} + \frac{1}{g\rho^2}\frac{d^2 g}{d\phi^2} + \frac{1}{h}\frac{d^2 h}{dz^2} + \beta^2 = 0$$

Separating Constants

$$\frac{1}{h}\frac{d^2 h}{dz^2} + \beta^2 = 0$$

$$\frac{d^2 h}{dz^2} + h\beta^2 = 0$$

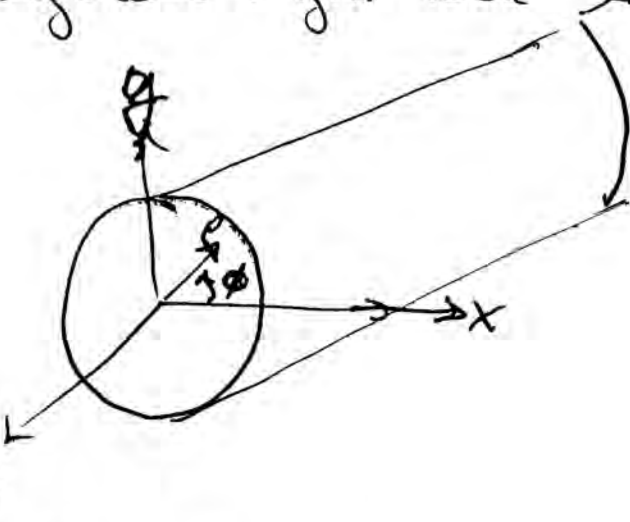
• Solutions to the Bessel Differential Equation

$$f_1(\rho) = A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)$$
$$f_2(\rho) = C_1 H_m^{(1)}(\beta_\rho \rho) + D_1 H_m^{(2)}(\beta_\rho \rho)$$
$$g_1(\phi) = A_2 e^{-jm\phi} + B_2 e^{+jm\phi}$$
$$g_2(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi)$$
$$h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}$$
$$h_2(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)$$

where J_m & Y_m are Bessel functions of 1st & 2nd kind

$H_m^{(1)}$ & $H_m^{(2)}$ are Hankel functions of 1st & 2nd kind

A typical solution for a cylindrical waveguide might look like



$$E(\rho,\phi,z) = \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]$$
$$\times \left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right]$$

Two types of fiber of interest

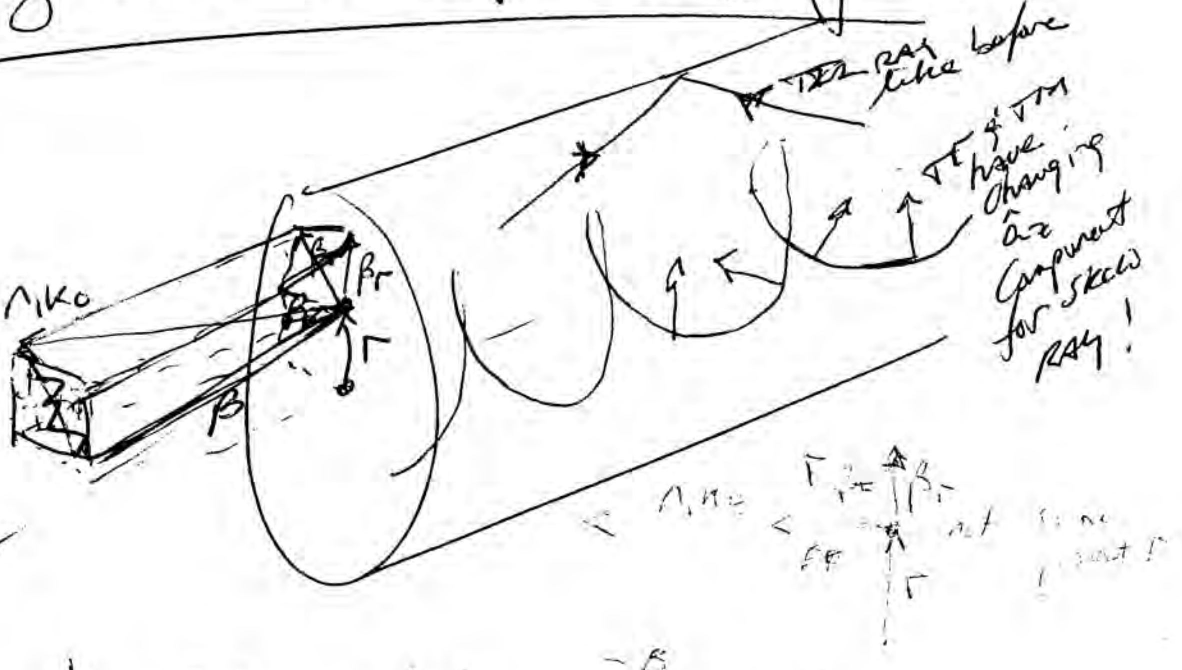
18/2

- Single Mode: only 1 mode can prop.: $2a \sim 6-10\mu m$ @ $1.3\mu m\ \lambda_o$
- Multi Mode: more than 1 mode can prop.: $2a \sim 50-100\mu m$ @ $1.3\mu m\ \lambda_o$

. Δ typically 0.2% SM

1.0% MM

Ray propagation in step index fiber



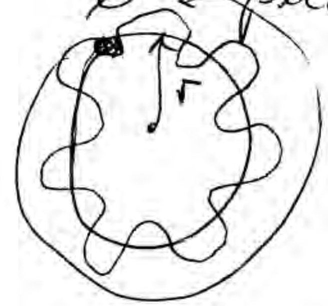
$k_1 = n_1 k_0$ as before

where $\beta_r^2 + \beta_\phi^2 + \beta^2 = n_1^2 k_0^2$

Transverse propagation constant: $\beta_t^2 = \beta_r^2 + \beta_\phi^2$

ϕ provides measure of "skewness"

We will See that a Contraint on Skew is Such that



st. Circumference

$$\beta_\phi 2\pi r = q\, 2\pi$$

$$\boxed{\beta_\phi = \frac{q}{r}}$$

Recall β_ϕ is also in r/m !

· z Component ⇒ β

· Modes ⇒ transverse Resonance established.

Transverse Resonance ⇔ Start & finish pts. Of Ray path with Same r & ϕ have Same phase.

·

3.2 Field Analysis of the Step index Fiber: (4)

Using Cylindrical Coordinate System lets write the E & H fields as

$$E = E_0(r,\phi)\, e^{-j\beta z} \qquad H = H_0(r,\phi)\, e^{-j\beta z}$$

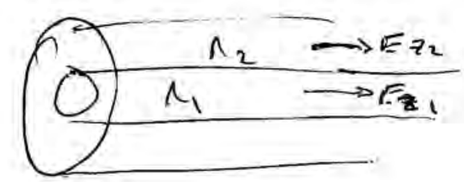
Mode profile

So the Mode profile or transverse patterns that we want to solve for are $E_0(r,\phi)$ and $H_0(r,\phi)$

As we showed for the Slab waveguide, the $\hat{z}$ Component of the fields obey the wave propagation eqns.

in the Core $\nabla_t^2 E_{z1} + (n_1^2 k_0^2 - \beta^2) E_{z1} = 0$ $r<a$

$r>a$ in the Cladding $\nabla_t^2 E_{z2} + (n_2^2 k_0^2 - \beta^2) E_{z2} = 0$ $r>a$



Lets define
$$\beta_{t_1}^2 = n_1^2 k_0^2 - \beta^2$$
$$\beta_{t_2}^2 = n_2^2 k_0^2 - \beta^2$$

Wave Eqn in Cylindrical Coord.

~~Then~~ a general wave equation can be written using the cylindrical version of ∇

$$\nabla = \frac{1}{r}\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

And

$$\nabla^2 E = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E}{\partial \phi^2} + \frac{\partial^2 E}{\partial z^2}$$

So the wave Eqn for $E_{z_{1,2}}$ in cylindrical Coordinates is

$$\frac{\partial^2 E_{z_{1,2}}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{z_{1,2}}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_{z_{1,2}}}{\partial \phi^2} + \beta_t^2 E_z = 0$$

- Seperable Solutions

Lets assume that the Solutions are "Seperable" in $r, \phi,$ & z as

$$E_i = R_i(r)\,\Phi_i(\phi)\,e^{-j\beta z}$$

- then the Complete Solution to the wave eqn is given by the Sum of all possible Solutions of the form

$$E_z = \sum_i R_i(r)\,\Phi_i(\phi)\,e^{-j\beta_i z}$$

where the subscript "i" denotes a particular mode.

⇒ Substituting any particular mode into the wave eqn.

$$\frac{\partial^2 R\Phi e^{-j\beta z}}{\partial r^2} + \frac{1}{r}\frac{\partial R\Phi e^{-j\beta z}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R\Phi e^{-j\beta z}}{\partial \phi^2} + \beta_t^2 R\Phi e^{-j\beta z} = 0$$

$$\Phi e^{-j\beta z}\frac{\partial^2 R}{\partial r^2}+\frac{\Phi e^{-j\beta z}}{r}\frac{\partial R}{\partial r}+\frac{R e^{-j\beta z}}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2}+\beta_t^2 R\Phi e^{-j\beta z}=0$$

multiplying through by r^2 and dividing through by $R\Phi$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+r^2\beta_t^2=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

where we have accumulated all r dependence on the LHS and Φ dependence on the RHS.

Note! r & ϕ can vary independently, therefore, in order for the above eqn. to hold, each side must be equal to the same constant! which we will arbitrarily choose as q^2, then

(i) $$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+q^2=0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+r^2\beta_t^2-q^2=0$$

(ii) or $$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left[\beta_t^2-\left(\frac{q}{r}\right)^2\right]R=0$$

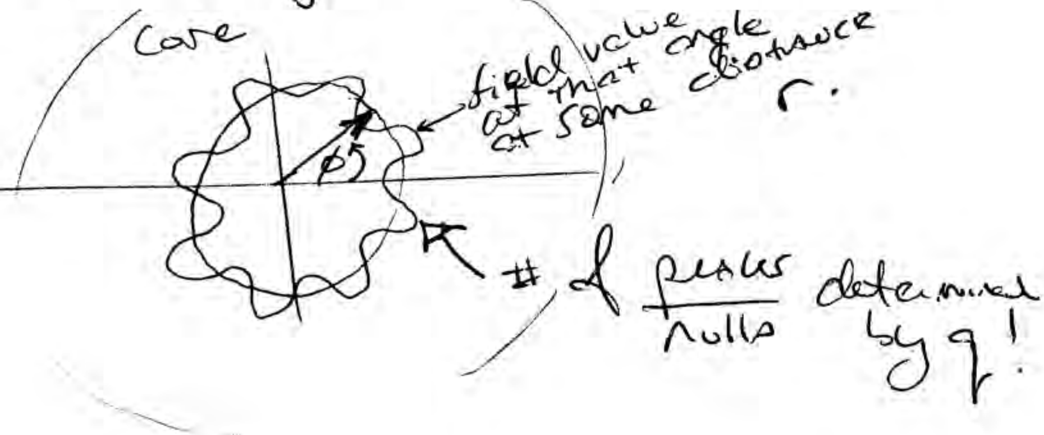
So now we have a wave Eqn. for Φ and one for R! the solutions to each are as follows

(i) let ~~Φ~~ =

$$\Phi(\phi) = \begin{cases} \cos(q\phi + \alpha) \\ \sin(q\phi + \alpha) \end{cases}$$

(α: constant phase shift)

Since $\Phi(\phi)$ must be self consistent with it self (i.e, repeat every 2π) q must be integer



(ii) this eqn. is of the form of Bessel's Eqn! It has solutions of the form

β_t [illegible]

$$R(r) = \begin{cases} A\,J_q(\beta_t r) + A'\,N_q(\beta_t R), & \beta_t \text{ real} \\ C\,K_q(|\beta_t| r) + C'\,I_q(|\beta_t| r), & \beta_t \text{ Im} \end{cases}$$

where we have defined

8

$$\beta_\phi = \frac{q}{r}$$

$$\beta_t^2 = \beta_r^2 + \beta_\phi^2$$

<u>Bessel functions</u> (See App. A for detail)

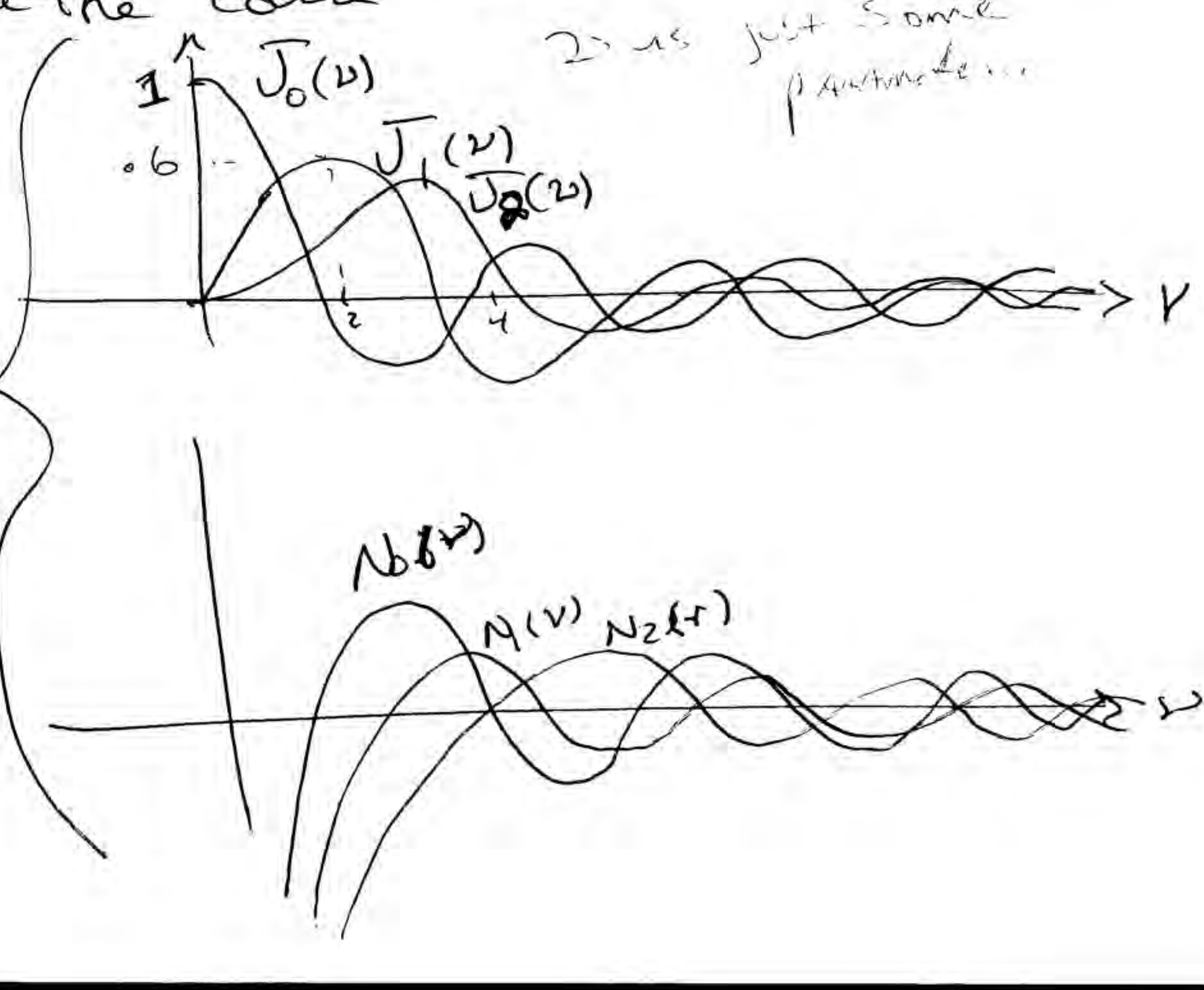
J_q & N_q : ordinary Bessel fnctns. of 1^{st} & 2^{nd} kind

K_q & I_q : modified Bessel fnctns of 1^{st} & 2^{nd} kind.

What do these functions look like?

Inside the core

$r \leq a$



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But this shouldn't be surprising, since we expect oscillatory solutions in the transverse direction "r"

$\Rightarrow$ But which ones are physically real?

Condition I the field should be finite at all points w/no singularities at $r=0$

$\Rightarrow$ $N_q(\beta_t r)$ does not work

So $A'=0$

$\Rightarrow$ Our solutions only can look like $J_q(\beta_t r)$

Also, since this solution applys to β_t real,

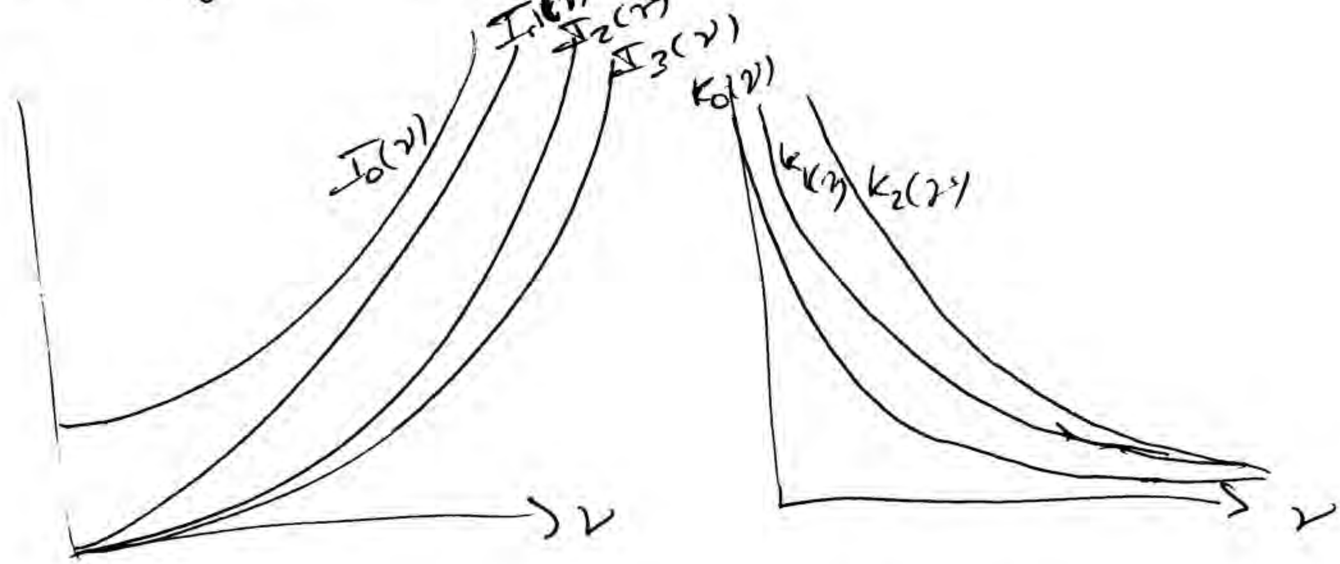
$$\beta_{t_1} = (n_1^2 k_0^2 - \beta^2)^{1/2} \text{ is real}$$

$\Rightarrow$ just as with slab!

remember β is really β_z!

Condition II

- Now lets look at the modified Bessel functions ; Looks like decaying exp.

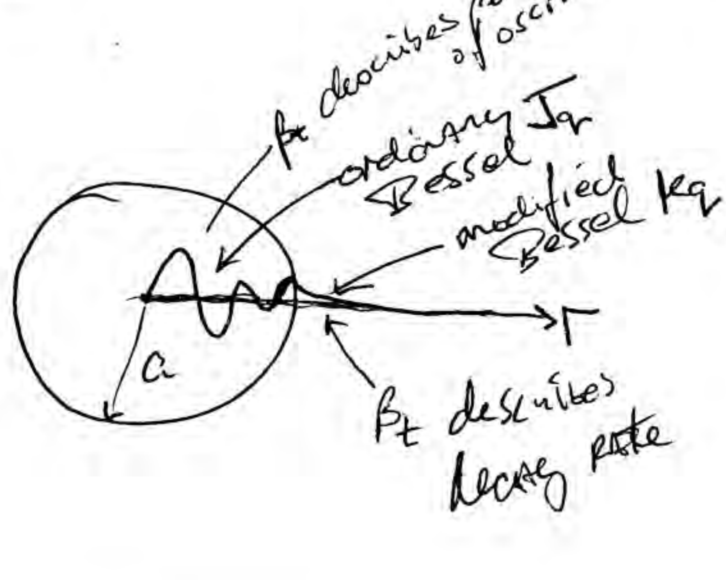


- this is the Solution in the cladding $r > a$, So decay would be for a guided mode

$$\Rightarrow K_0(\nu) \text{ Only}$$

$$\Rightarrow C' = 0 \,! \quad \Rightarrow \beta_{t2} \text{ imag.}$$

For Example :



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Normalized Parameters

$$u = \beta_{t1} a = a\left(n_1^2 k_0^2 - \beta^2\right)^{1/2}$$

$$w = \left|\beta_{t2}\right| a = a\left(\beta^2 - n_2^2 k_0^2\right)^{1/2}$$

note we use magnitude!

units are $\beta_t a = \left(\frac{r}{m}\right) m = \text{rad}$

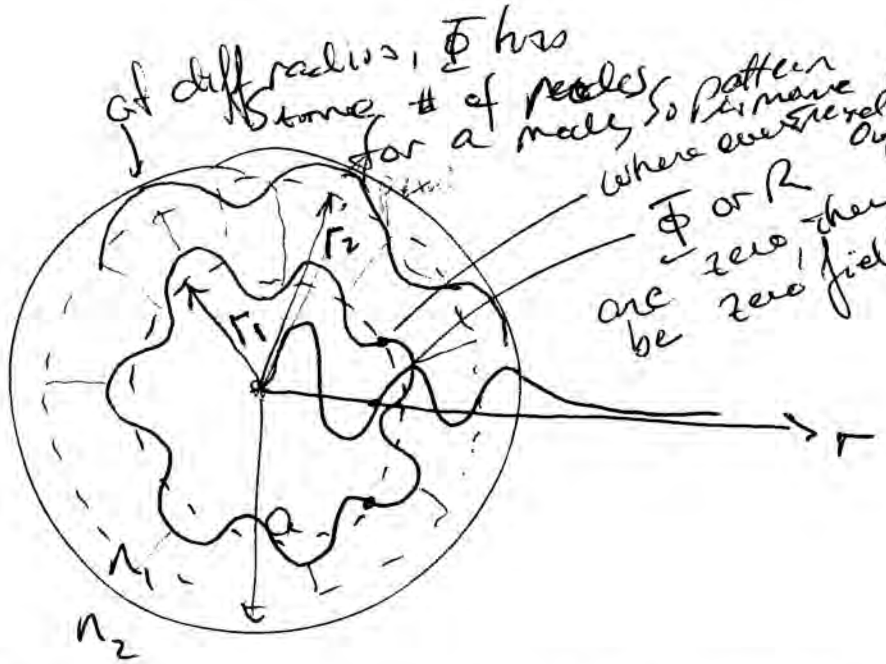
$\Rightarrow$ Setting $A' = C' = 0$ and $\alpha = 0$

then $\boxed{E = R(r)\,\Phi(\phi)\,e^{-j\beta z}}$ Separable Solution

can be written inside & outside core

$$E_z = \begin{cases} A\, J_q\left(\frac{ur}{a}\right)\sin(q\phi)\,e^{-j\beta z}, & r \le a \\ C\, K_q\left(\frac{wr}{a}\right)\sin(q\phi)\,e^{-j\beta z}, & r \ge a \end{cases}$$

What do these modes look like?



Φ & R modulate each other in circumference & radial direction respectively.

Magnetic field

The wave Eqn. for H can be solved to yield the following solutions

$$H_z = \begin{cases} B\, J_q(\mu r/a)\cos(q\phi)\, e^{-i\beta z}, & r \le a \\ D\, K_q\left(\frac{\omega r}{a}\right)\cos(q\phi)\, e^{-i\beta z}, & r \le a \end{cases}$$

the cos form was chosen so that the BC's can be satisfied.

Expansion of Maxwells in Cylindrical Coord. (13)

just as we expanded ME's into 6 Eqns for x, y, z, we can expand in r & ϕ

$$E_r = \frac{-j}{\beta_t^2}\left(\beta \frac{\partial E_z}{\partial r} + \omega\mu \frac{1}{r}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi = \frac{-j}{\beta_t^2}\left(\beta \frac{1}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu \frac{\partial H_z}{\partial r}\right)$$

$$H_r = -\frac{j}{\beta_t^2}\left(\beta \frac{\partial H_z}{\partial r} - \omega\epsilon \frac{1}{r}\frac{\partial E_z}{\partial \phi}\right)$$

$$H_\phi = \frac{-j}{\beta_t^2}\left(\beta \frac{1}{r}\frac{\partial H_z}{\partial \phi} + \omega\epsilon \frac{\partial E_z}{\partial r}\right)$$

Substituting 3.16 & 3.17 ⇒ leads to table 3.1

just as before, we substitute

just obtained these ⇒ E_z & H_z into these Eqns. to obtain all the fields

See Eqns. 3.22 - 3.33 in text

for $r<a$ $E_z, E_r, E_\phi, H_z, H_r, H_\phi$

$r<a$ $E_z, E_r, E_\phi, H_z, H_r, H_\phi$

Bessel function Relationship

From Eqns. 3.22-3.33 we used the following definition ~~property~~ of Bessel functns.

$$J_q' = \frac{d}{d\left(\frac{ur}{a}\right)}\left(J_q\left(\frac{ur}{a}\right)\right)$$

Mode Classification & Characteristic Eqn.

Lets Consider the lowest order mode, a special case from the 1st Example.

Let $q=0$ and $E_{z_{1,2}}=0$

then $A = C = 0$

And the nonzero fields are H_z, E_ϕ, H_r } from Eqns. TABLE 3.1

This looks like the transverse TE mode in a dielectric slab waveguide, and we denote it by

H_r, E_ϕ, H_z

$$TE_{q,m} = TE_{0m}$$

$q=0$

- We need to have both q & m

Since q defines the radial period via Bessel function

m defines k_x, β & γ_2!

or given $m \Leftrightarrow$ given ω, β.

- $q = 0$ $E_z = 0$
- When $q = 0$ & $H_z = 0$,

$TM_{0m} \Rightarrow E_z, E_r, H_\varphi$

$\{TE_{0m}$, TM_{0m}

- Since TE & TM describe transverse fields

i.e. $TE \Rightarrow E_z = 0$

$TM \Rightarrow H_z = 0$

there are no TE & TM for $q \neq 0$!

Since nulls & peaks

For $q \neq 0$
We need to talk about
* **Hybrid Modes**: 4 components of/ E & H

HE_{qm} & EH_{qm} $q \neq 0$

Continuity of tangent field components at $r = a$

For z - components: from Eqns 3.22–3.33

(i) $A J_q(u) - C K_q(w) = 0$ 3.22 & 3.28

for $E_{z_1}^{*}\big|_{r=a} - E_{z_2}^{*}\big|_{r=a} = 0$

(ii) $B J_q(u) - D K_q(w) = 0$ 3.25 & 3.31

$H_{z_1}(a) = H_{z_2}(a)$

For ϕ components at $r = a$! define this!

$$A \frac{j\beta}{(u/a)^2} \frac{q J_q(u)}{a} - B \frac{j\omega\mu}{(u/a)} J_q'(u) + C \frac{j\beta}{(w/a)^2} \frac{q K_q(w)}{a} - D \frac{j\omega\mu}{(w/a)} K_q'(w) = 0 \quad \text{(iii)}$$

$E_{\phi_1}(a) = E_{\phi_2}(a)$

$$A \frac{j\omega\epsilon_0 n_1^2}{(u/a)} J_q'(u) - B \frac{j\beta}{(u/a)^2} \frac{q J_q(u)}{a} + C \frac{j\omega\epsilon_0 n_2^2}{(w/a)} K_q'(w) - D \frac{j\beta}{(w/a)^2} \frac{q K_q(w)}{a} = 0 \quad \text{(iv)}$$

$H_{\phi_1}(a) = H_{\phi_2}(a)$

(i) – (iv) can be written in matrix form.

writing out all Boundary Conditions as a system of Homogeneous Equations to be Solved Simultaneously

$$[M]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

Set of Homogeneous diff equs.

Coefficients of A, B, C, D respectively

$$M = \begin{bmatrix} J_q(u) & 0 & -K_q(w) & 0 \\ 0 & J_q(u) & 0 & -K_q(w) \\ \frac{j\beta}{(u/a)^2}\frac{q\,J_q(u)}{a} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- A solution exists if and only if M is in-vertable (i.e. if M^{-1} exists) and this only holds true if the $\det[M] = 0$!

Characteristic Eqn (Eigenvalue Eqn) (18)

- Evaluating $\det[M] = 0$

(I) $$\left[\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(\omega)}{\omega K_q(\omega)}\right]\left[\frac{n_1^2}{n_2^2}\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(\omega)}{\omega K_q(\omega)}\right]$$

$$= q^2\left(\frac{1}{u^2} + \frac{1}{\omega^2}\right)\left(\frac{n_1^2}{n_2^2}\frac{1}{u^2} + \frac{1}{\omega^2}\right)$$

Now lets look at weakly guiding case:
In the weakly guiding approximation

$$n_1 \simeq n_2$$

- and

(II) $$\left[\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(\omega)}{\omega K_q(\omega)}\right] = \pm q\left(\frac{1}{u^2} + \frac{1}{\omega^2}\right)$$

Recall $\Rightarrow$ $$\left.\begin{aligned} u &= a\left(n_1^2 k_0^2 - \beta^2\right)^{1/2} \\ \omega &= a\left(\beta^2 - n_2^2 k_0^2\right)^{1/2} \end{aligned}\right\} n_1, n_2, k_0$$

So β is completely defined by physical parameters a, n_1, n_2 & k_0 or λ_0!
in order to satisfy this equation
- for a particular q!

* ~~For $q>0$ contd.~~ (20)

From Buck, eqn A.19 $\Rightarrow F_1'(\beta_t r) = F_0(\beta_t r) - \frac{1}{\beta_t r} F_1(\beta_t r)$

using identities

$$J_\ell'(x) = \pm J_{\ell \mp 1}(x) \mp \ell \frac{J_\ell(x)}{x}$$

$$K_\ell'(x) = -K_{\ell \mp 1}(x) \mp \ell \frac{K_\ell(x)}{x}$$

* For $q=0$, ... TM_{om}

we can write the characteristic Eqn. as

for $q>0$

$$\frac{J_{q-1}(u) - q\frac{J_q(u)}{u}}{u J_q(u)} + \frac{-K_{q-1} - q\frac{K_q(\omega)}{\omega}}{\omega K_q(\omega)} = \ldots$$

and $+q\left(\frac{1}{u^2}+\frac{1}{\omega^2}\right)$

$$\frac{J_{q-1}}{uJ_q} - \frac{q}{u^2} - \frac{K_{q-1}}{\omega K_q} - \frac{q}{\omega^2} = -q\left(\frac{1}{u^2}+\frac{1}{\omega^2}\right)$$

$$\frac{J_{q-1}}{uJ_q} - \frac{K_{q-1}}{\omega K_q} = 0$$

$$\frac{J_{q-1}}{uJ_q} = \frac{K_{q-1}}{\omega K_q}$$

$(q>0)$

$$\boxed{\frac{uJ_q}{J_{q-1}} = \frac{\omega K_q}{K_{q-1}}}$$

weakly guiding $\Rightarrow$ ~~EH~~$_{qm}$ HE_{qm}

Similarly, for $-q\left(\frac{1}{u^2}+\frac{1}{w^2}\right)$

$$\boxed{u\frac{J_q}{J_{q+1}} = -w\frac{k_q}{k_{q+1}} \Rightarrow EH_{qm}}$$

~~recurrence~~ EH Eqn as

$$\cancel{u\frac{J_{q-2}}{J_{q-1}} = }$$

The HE, EH & TE & TM modes
Can All Combine to Create
LP_{lm} modes
which we will discuss later.

~~do this pre next~~

Solutions to Eigenvalue (characteristic) Eqns.

graphical Solution ⇒ Similar to Slab Waveguide

Define: Normalized freq parameter

like R! $V = (u^2+w^2)^{1/2}$

$= ak_0(n_1^2-n_2^2)^{1/2}$

$= n_1 a k_0\sqrt{2\Delta}$

V = V Number! Very Important

(22)

V-number: fiber light freq

$$V(n_1, n_2, a, \lambda_0) \Rightarrow \text{physical parameters}$$

V is like R for slab waveguide

· Look at $q=0$ first

E_r, E_z, H_ϕ TM_{om}

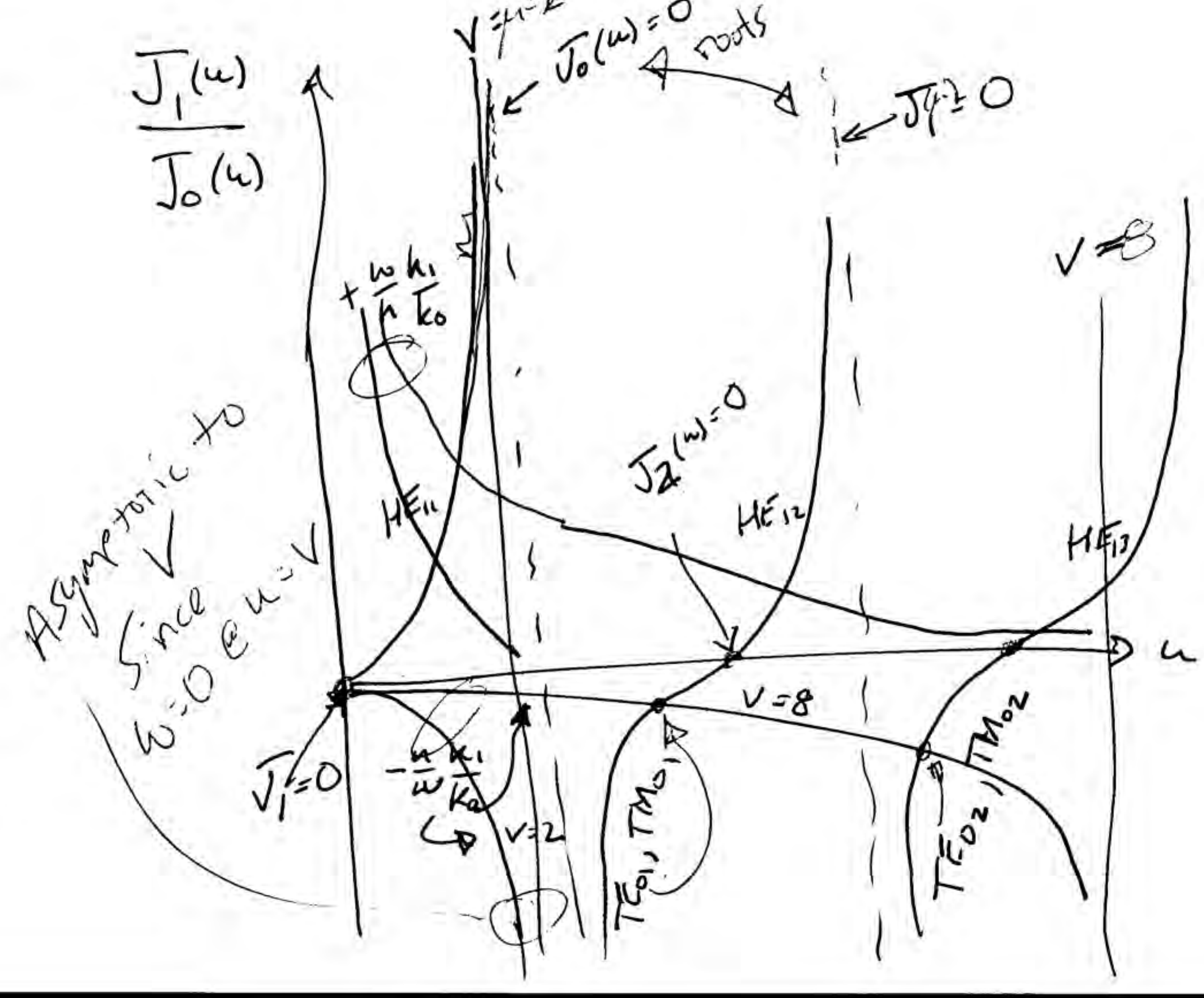
H_r, H_z, E_ϕ TE_{om}

$$\frac{J_1(u)}{J_0(u)} = -\frac{u}{\omega}\,\frac{K_1(\omega)}{K_0(\omega)},\quad TE_{om}$$

$$\frac{J_1(u)}{J_0(u)} = -\frac{n_2^2}{n_1^2}\,\frac{u}{\omega}\,\frac{K_1(\omega)}{K_0(\omega)}\quad TM_{om}$$

J_0 pass through zero with period that produces asymptotes for the LHS

due to zeroes



Consider only $0 \le \mu \le V$ for decaying evanescent fields! (23)

So we see that for

$$w = 0, \quad V = u \,!$$

And asymptotes for $\frac{w}{u}\frac{K_1}{K_0}$ & $-\frac{w}{w}\frac{K_1}{K_0}$ are at V!

<u>Note</u>: Upper plane curves are for HE modes

Since $\cancel{\frac{J_{q-1}}{J_q} = \frac{u}{w}\frac{K_{q-1}}{K_q}}$

$l = q+1$

$$\frac{J_1(u)}{J_0(u)} = -\frac{w K_1(w)}{u K_0(w)} \qquad \frac{J_q}{J_{q+1}} = \frac{w}{u}\frac{K_q}{K_{q+1}}$$

we show $HE_{qm} = HE_{1m}$

Lower plane curves are for

$$\cancel{\frac{J_q}{J_{q+1}}} \quad \frac{J_{q+1}}{J_q} = -\frac{u}{w}\frac{K_{q+1}}{K_q}$$

we show TE_{0m}, TM_{0m} modes!

Major points to note:

- Solutions to characteristic Eqns are at intersecting points & changes with V
- Knowing V and u's ⇒ w & β knowing V!
- Number of propagating modes is equal to number of zeros in $J_1(u)$ less than $u = V$.
- As V increases, vertical line moves out and more modes propagate.
- If V is a zero of $\frac{J_1}{J_0}$ curve, that HE mode of branch mode is cutoff.
- If V is at a lower asymptote, cutoff is reached for TE or TM mode of that branch
- HE_{11} has no cutoff and single mode operation is only propagation of HE_{11} which occurs for $0 < V < 2.405$!

Single mode

Important!

Field functions

→ Solving for E & H fields (see notes)

TE_{0m}, TM_{0m}

Setting $q=0$

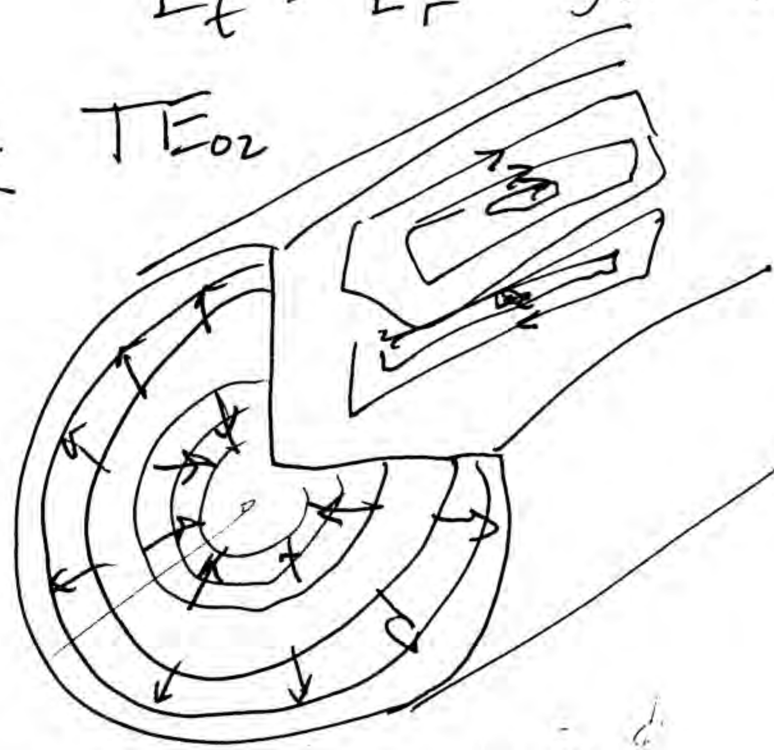
$$\begin{cases} E_t\,(r<a) = E_0\, J_1\left(\frac{ur}{a}\right) \\ E_t\,(r>a) = -E_0\,\frac{u}{w}\,\frac{J_0(u)}{K_0(w)}\,K_1\left(\frac{wr}{a}\right) \end{cases}$$

$E_t = E_\phi$ for TE

$E_t = E_r$ for TM

Example: TE_{02}

$q=0$
$m=2$



m=2 = # half cycles

no periodicity in ϕ

Mode Propagation Constant

recall $V^2 = U^2 + W^2$

let $W \approx (V^2 - U_c^2)^{1/2} = V\left(1 - \frac{U^2}{V^2}\right)^{1/2}$

where U_c is U @ cutoff for a particular mode

Define: normalized propagation constant

$$b = 1 - \frac{U^2}{V^2} = \frac{\beta^2 - n_2^2 k_0^2}{n_1^2 k_0^2 - n_2^2 k_0^2} = \frac{n_{eff}^2 - n_2^2}{n_1^2 - n_2^2}$$

$$\approx \frac{n_{eff} - n_2}{n_1 - n_2} \quad (\text{for } n_1 \approx n_2)$$

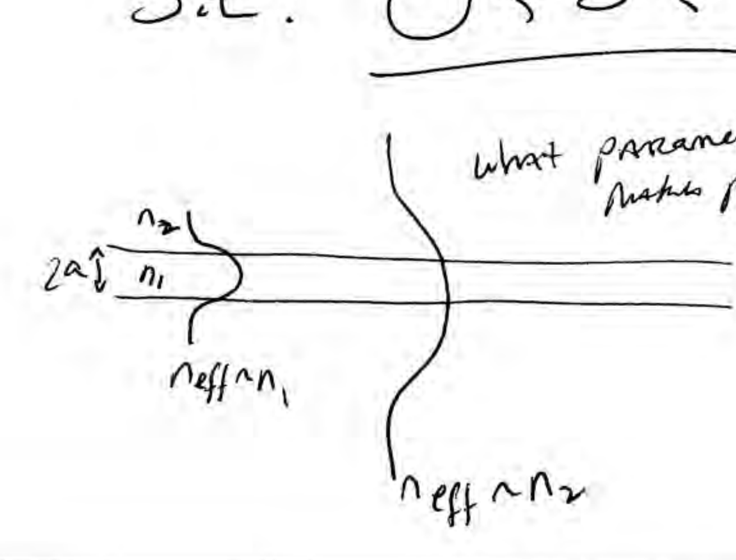
where $n_{eff} = \frac{\beta}{k_0}$

note that n_{eff} is the effective index that the particular mode "sees"

$n_2 < n_{eff} < n_1$

s.t. $0 < b < 1$

what parameters makes profile change?



Linear Polarized Modes

Superposition of propagating modes w. V #.
Where superposition is polarized in one cartesian direction.

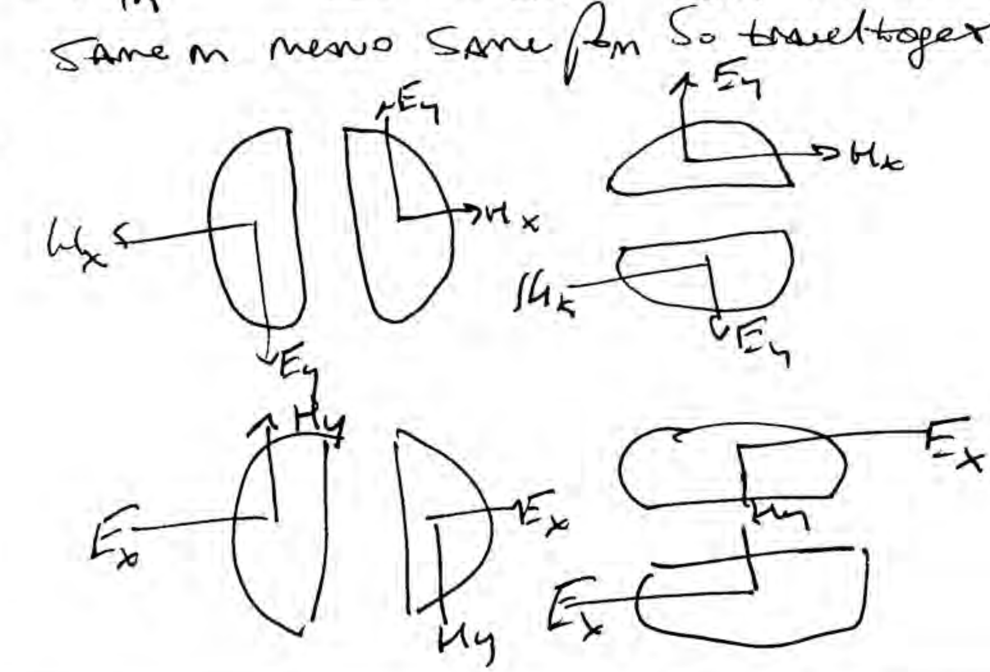
V	l	degen modes	LP mode
0	0	HE_{11}	LP_{01}
2.405	1	$\underbrace{TE_{01}, TM_{01}, HE_{21}}_{\text{same m}}$	LP_{11}
3.832	2	EH_{11}, HE_{31}	LP_{21}
3.832	0	HE_{12}	LP_{02}

~~Coupling~~ LP_{lm}

---

note: $LP_{lm} = \sum HE_{2m}\ TE_{0m}\ TM_{0m}$
same m means same β_m so travel together!

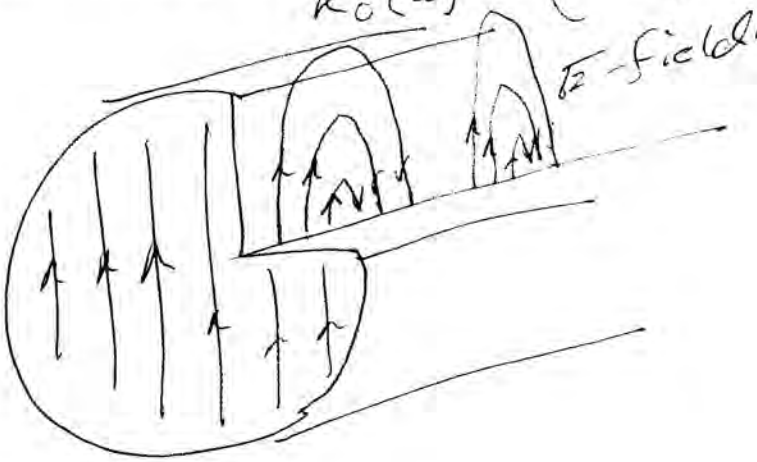
$LP_{11} \Rightarrow$



for LP_{0m} ⇒ only HH_{1m} is present.

$$E_{LP_{0m}} = E_0 J_0\left(\frac{ur}{a}\right) \quad r \leq a$$

$$E_{LP_{0m}} = E_0 \frac{J_0(u)}{K_0(\omega)} K_0\left(\frac{\omega r}{a}\right) \quad r \geq a$$

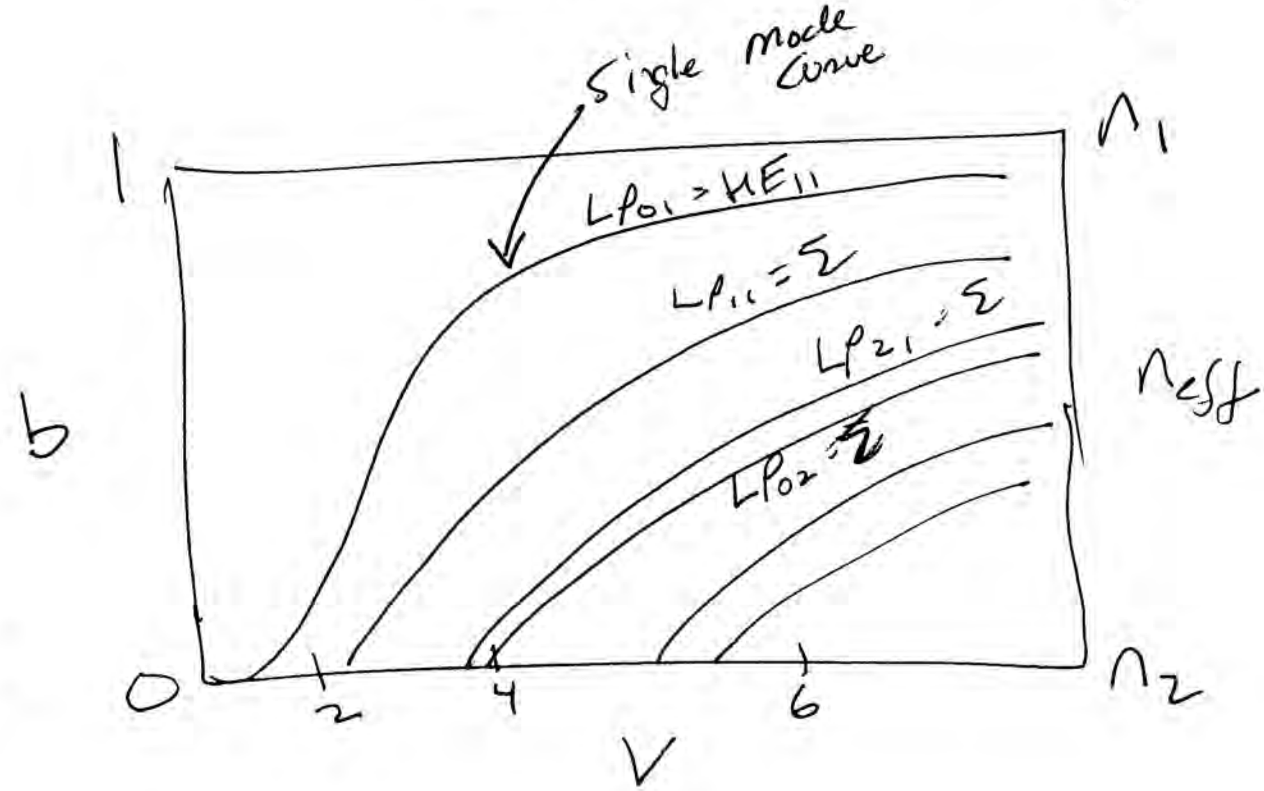


Experimentally ⇒ LP modes will show as intensity patterns.

See LP mode profiles in Fig 3.5

* l ⇒ $\frac{1}{2}$ # min or max in intensity profile as ϕ $0 \to 2\pi$
* m ⇒ # maxima radially

A very useful plot



discuss a lot!

For the range $1.3 < V < 3.5$

$$b \approx \frac{(1.1428V - 0.9960)^2}{V^2}$$

Accurate to within .1% for $1.5 < V < 2.4$

i.e. ⇒ for Multimode design!

⇒ Very important for dispersion

Fiber Mode Review

- We solved for the fiber characteristic eqn.

$$\left[\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(w)}{w K_q(w)}\right]\left[\frac{n_1^2}{n_2^2}\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(w)}{w K_q'(w)}\right]$$
$$= q^2\left(\frac{1}{u^2}+\frac{1}{w^2}\right)\left(\frac{n_1^2}{n_2^2}\frac{1}{u^2}+\frac{1}{w^2}\right)$$

- which for weakly guiding case $n_1 \simeq n_2$ reduces to

$$\left[\frac{J_q'(u)}{u J_q(u)} + \frac{K_q'(w)}{w K_q(w)}\right] = \pm q\left(\frac{1}{u^2}+\frac{1}{w^2}\right)$$

where
$$u = a\left(n_1^2 k_0^2 - \beta^2\right)^{1/2}$$
$$w = a\left(\beta^2 - n_2^2 k_0^2\right)^{1/2}$$

- Given a set of physical parameters

$$k_0, a, n_1, n_2$$

And a value for q, yields a set of Solutions

$$\{\beta_q\}$$

To denote each solution in this set we use the subscript m

$$\boxed{\beta_{qm}}$$

<u>Modes</u>

Each β_{qm} corresponds to one mode of propagation

· Generally both E_z & H_z are non zero

$\Rightarrow$ Exception $q = 0$ $\begin{cases} E_z = 0 \\ \text{OR} \\ H_z = 0 \end{cases}$

So we call these TE_{om} and TM_{om}
HE_{om} EH_{om}

· Hybrid modes for $q \neq 0$

$$E_z < H_z \Rightarrow EH_{qm}$$

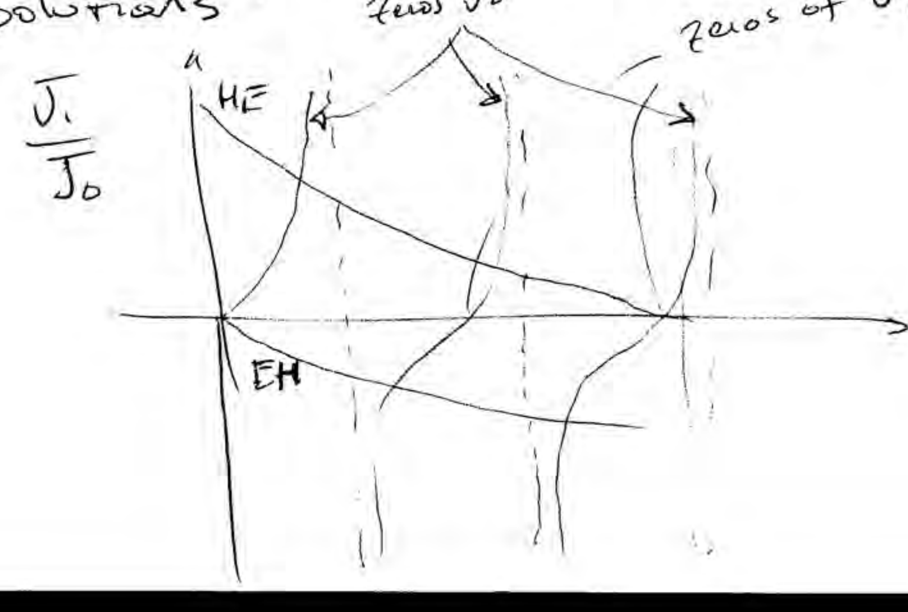
$$H_z < E_z \Rightarrow HE_{qm}$$

· Next we ~~plot the~~ seperate the HE & EH Solutions in the characteristic Equation

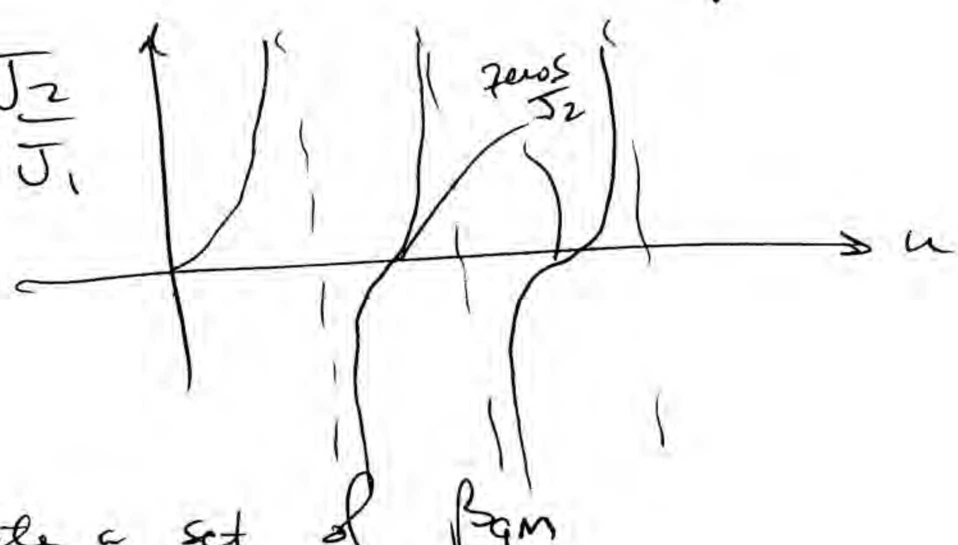
$$\frac{J_q}{J_{q+1}} = \frac{w}{u} \frac{K_q}{K_{q+1}} \quad \text{HE modes}$$

$$\frac{J_{q+1}}{J_q} = -\frac{u}{w} \frac{K_{q+1}}{K_q} \quad \text{EH modes}$$

· we plotted the RHS as a function of u and the left hand side as a function of u for $q = 0$ to generate a set of M Solutions



- The HE modes appear in the upper plane
  the EH modes appear in the lower plane
  - the upper plane HE curve terminates at zero at $u=V$
    the lower plane EH curve is Asymptotic at $u=V$
  - the number of modes that propagate for $q=0$ are given by the number of intersections M
  - the # of intersections can be changed by adjusting the physical parameters or "V"
- We can do the same for $q=1$



and generate a set of β_{qm}

- All β_{qm} elements that are equal are <u>degenerate</u> according to

$$l = \begin{cases} 1 & q=0 \\ q+1 & EH_{qm} \\ q-1 & HE_{qm} \end{cases}$$

For Example

$$l = 1 \begin{cases} \rightarrow q = 0 & \text{TE, TM} \\ \rightarrow q = 2 & \text{HE} \end{cases}$$

$$l = 2 \begin{cases} \rightarrow q = 1 & \text{EH} \\ \rightarrow q = 3 & \text{HE} \end{cases}$$

Defined set of LP modes such that

L_{lm}

all modes with same l & same m

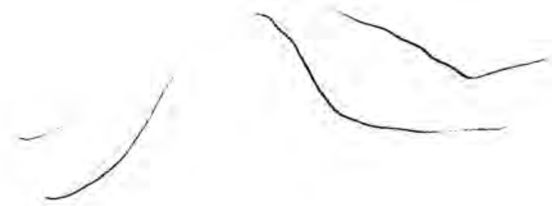
the intensity profile of LP modes
is proportional to $|E|^2$

$r \le a$ $\quad I_{lm} = I_0 J_l^2\left(\frac{ur}{a}\right)\sin^2(l\phi)$

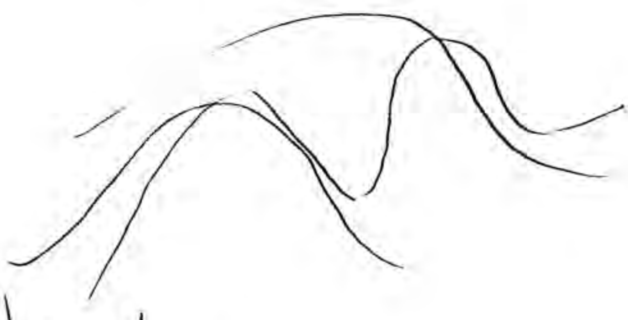
$r \ge a$ $\quad I_{lm} = I_0 \left(\frac{J_l(u)}{K_l(w)}\right)^2 K_l^2\left(\frac{wr}{a}\right)\sin^2(l\phi)$

when for $l=0$, $\sin(l\phi) \ne 0$
since really $\sin(l\phi + \alpha)$
So we just drop the sine function

LP_{01} $u=2$



LP_{11}



See text.

Cutoff wavelength

Since $V = a \frac{2\pi}{\lambda_0} (n_1^2 - n_2^2)^{1/2}$

we can define λ_c as the wavelength which satisfies the relationship for a V_c that is cutoff for a mode in question

$$\lambda_c = \frac{a 2\pi}{V_c} (n_1^2 - n_2^2)^{1/2}$$

So for single mode operation,

let $V_c = 2.405$

$$\lambda_c = a \frac{2\pi}{2.405} (n_1^2 - n_2^2)^{1/2}$$

$$\lambda_c = \left(\frac{2\pi}{2.405}\right) a n_1 \sqrt{2\Delta}$$

- From now on we will talk primarily about single mode operation

$$V < 2.405$$

the LP_{01} mode is the only mode that propagates, and it can be approximated by a "Gaussian" intensity profile.

What does ~~β_{lm}~~ ~~look~~ for each mode?

$$u = a\left(n_1^2 k_0^2 - \beta^2\right)^{1/2}$$

$$b = \frac{\beta^2 - n_2^2 k_0^2}{n_1^2 k_0^2 - n_2^2 k_0^2}$$

$$\beta_{lm}^2 = k_0^2 b\left(n_1^2 - n_2^2\right) + n_2^2 k_0^2$$

$$= k_0^2\left(\frac{n_2^2}{2n_1^2} + \frac{b\left(n_1^2 - n_2^2\right)}{2n_1^2}\right) 2n_1^2$$

$$= k_0^2 \cancel{2} n_1^2\left(\frac{n_2^2}{\cancel{2}n_1^2} + 2b\Delta\right)$$

$$\beta_{lm} = n_1 k_0 \left(\frac{n_2^2}{n_1^2} + 2b\Delta\right)^{1/2}$$

$$\simeq n_1 k_0 \left(1 + 2b\Delta\right)^{1/2} \quad \text{for } n_1 \sim n_2$$

$$\boxed{\beta_{lm} \cong n_1 k_0 \left(1 + b\Delta\right) \quad \text{for } 2b\Delta \ll 1}$$

where we have defined

$$\beta_\phi = \frac{q}{r}$$

$$\beta_t^2 = \beta_r^2 + \beta_\phi^2$$

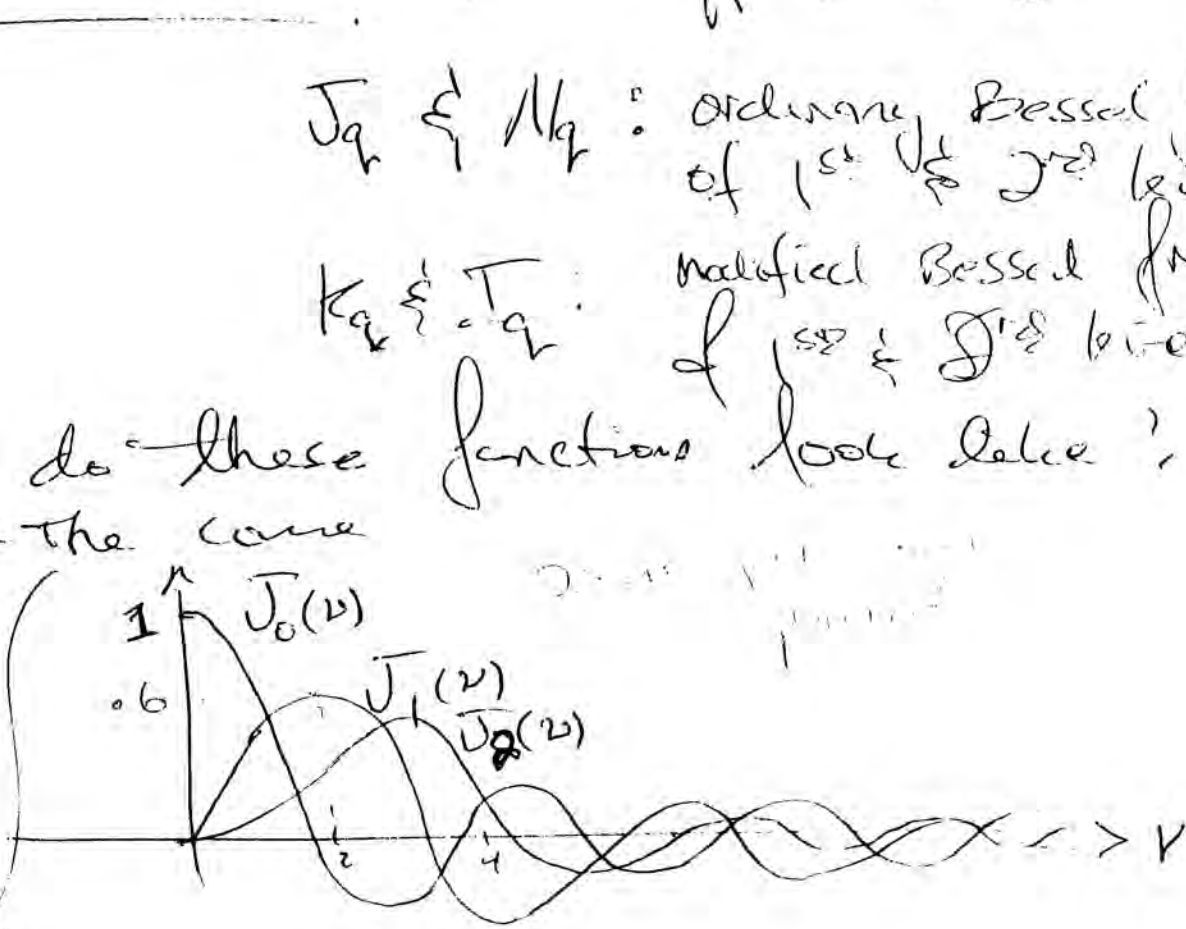
Bessel functions (See App. A for detail)

J_q & N_q : ordinary Bessel fnctns. of 1st & 2nd kind

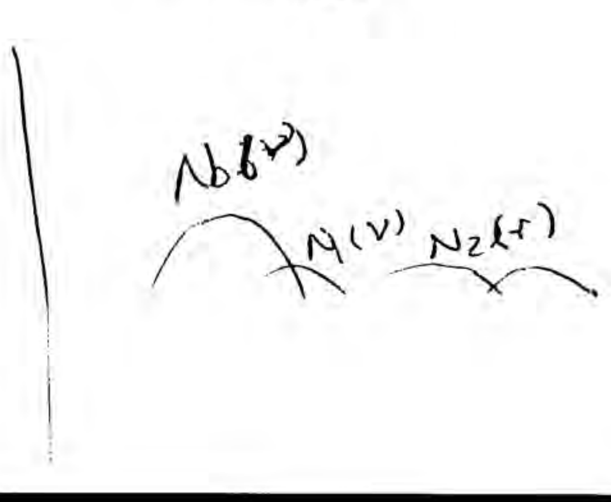
K_q & I_q : modified Bessel fnctns of 1st & 2nd kind

What do these functions look like?

Inside the core



$r \le a$



But this shouldn't be surprising, since we expect oscillatory solutions in the transverse direction "r"

⇒ But which ones are physically real?

Condition I: the field should be finite at all points w/no singularities at r = 0

⇒ $N_q(\beta_t r)$ does not work

So $\boxed{A' = 0}$

⇒ Our solutions only can look like $J_q(\beta_t r)$

Also, since this solution applys to β_t real,

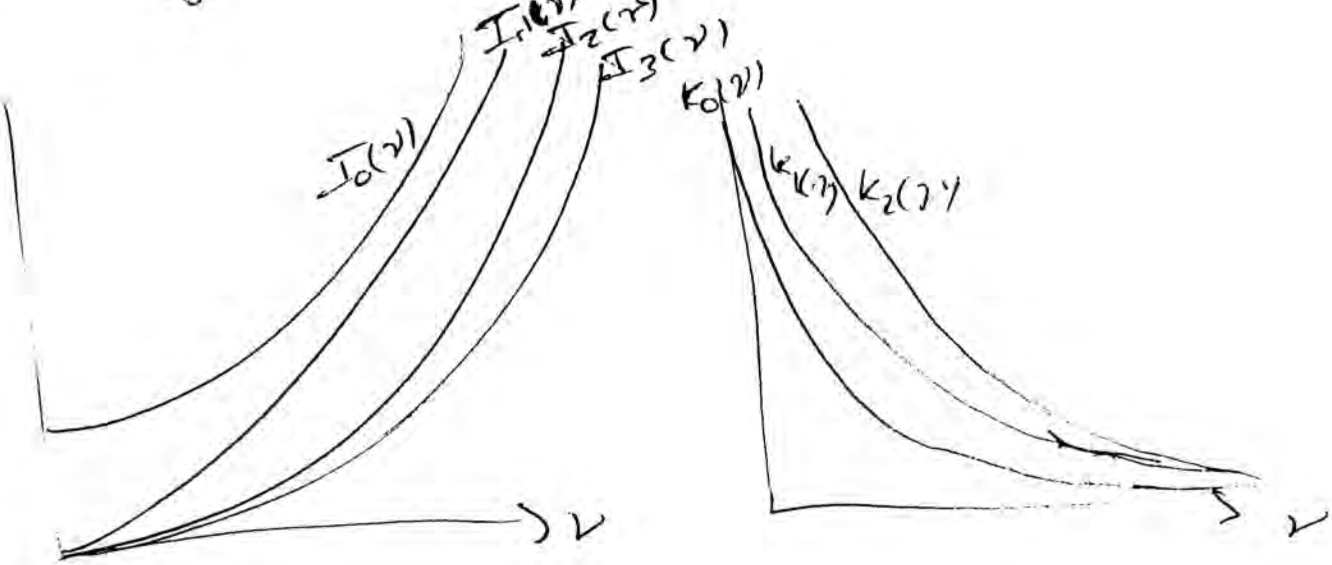
$$\beta_{t_1} = (n_1^2 k_0^2 - \beta^2)^{1/2} \text{ is real}$$

⇒ just as with Slab!

remember β is really β_z!

Condition II

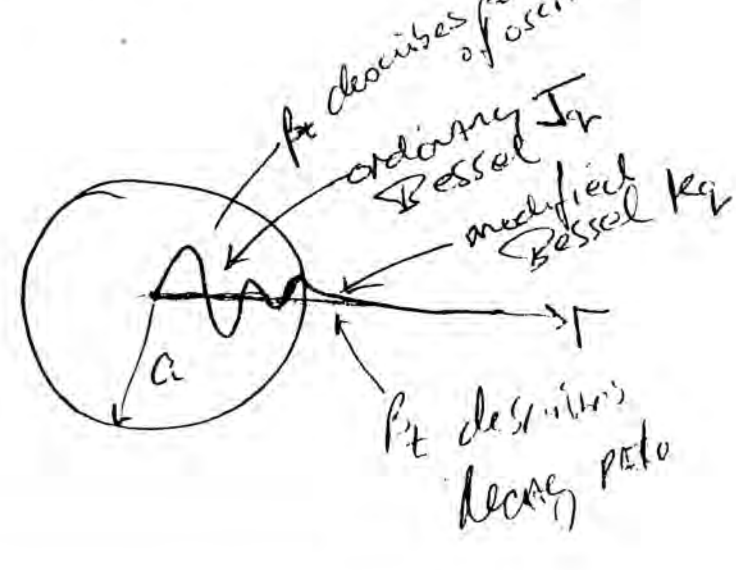
- Now lets look at the modified Bessel functions



- this is the Solution in the cladding $r > a$, So decay would be for a guided mode

$$\Rightarrow K_0(\nu) \text{ only}$$
$$\Rightarrow C' = 0 ! \quad \Rightarrow \beta_{t_2} \text{ imag.}$$

For Example:



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