UCSB Fall 2009

ECE 235: Problem Set 5

Assigned: Friday November 13
Due: Tuesday November 24 (by noon, in course homework box)
Reading: Hajek, Chapter 4 and selected parts of Chapter 6; class notes
Topics: Independent increments processes; Markov processes

Practice problems (not to be turned in): The even numbered problems in Chapter 4, up to Problem 4.34. Ignore all problems (or parts of problems) related to martingales, since we have not covered these. Compare your solutions with the solutions provided to make sure you understand the concepts.

Problem 1: Consider the random walk $X_n = W_1 + ... + W_n + X_0$, with $\{W_k\}$ i.i.d. with $P[W_k = +1] = p = 1 - P[W_k = -1]$. Set $X_0 = 0$. Suppose that $p < 1/2$, so that the random walk is drifting downwards. We want to find the probability that the random walk hits a positive level (which should become more and more unlikely, the higher the level).
(a) For any positive integer k, let $p_k = P[max_{n\geq 1} X_n \geq k]$. Show that $p_k = p_1^k$.
Hint: Given that the random walk ever reaches the value 1, consider a new random walk starting at that time.
(b) Write an equation for p_1 by conditioning on the value of X_1.
(c) Solve for p_1, and hence p_k.
(d) Show that your answer is $\exp(-\theta k)$, and specify the numerical value of θ for $p = \frac{1}{4}$.

Problem 2: Consider a random walk as in Problem 1, with $p = \frac{1}{4}$.
(a) Find $P[X_{18} = 10 | X_3 = 3]$.
(b) Find $P[X_{19} = 10 | X_3 = 3]$.
(c) Find the number of paths that go from $X_3 = 3$ to $X_{18} = 10$ without hitting 0 in between.
(d) Find the probability of going from $X_3 = 3$ to $X_{18} = 10$ without hitting 0 in between.

Problem 3: Let $\{W_t, t \geq 0\}$ denote a Wiener process such that $\mathbb{E}[W_2^2] = 8$.
(a) Find $P[W_4 > 8 | W_2 = -1, W_5 = 7]$.
(b) Specify the distribution of $W_4 - 2W_2$.
(c) Define the process $X_t = tW_{1/t}$, $t \geq 0$ (note that $X_0 = 0$ in the mean squared sense, let us just set $X_0 = 0$.). Provide a complete statistical description of X. Do you recognize it as one of the processes we have discussed in class?

Problem 4: Let $\{N_t, t \geq 0\}$ denote a Poisson process of rate λ. Let $\{U_k\}$ denote the inter-count intervals, as in the text.
(a) Find $P[N_{0.5} = 1, N_{2.5} = 3, N_{5.5} = 5]$.
(b) Find $P[N_{2.5} = 3 | N_{0.5} = 1, N_{5.5} = 5]$.
(c) Find an explicit expression for the CDF of the time of the kth arrival, $T_k = U_1 + ... + U_k$, which involves *only summation of a finite number of terms, and no integration*:

$$P[T_k = U_1 + ...U_k \leq x]$$

Hint: Relate this probability to a probability involving a sample of the Poisson process N.

Problem 5: Problem 4.31.

Problem 6: Consider the Markov chain of Problem 4.31.
(a) Find the equilibrium distribution.
(b) Suppose we start from state 1 at time 0. What is the probability that we are at state 3 at time 5? (You may want to use a computer to do this.)

Problem 7: Calls in a telephone exchange arrive according to a Poisson process of rate λ. Call durations are modeled as iid exponential random variables with parameter μ. The exchange can support a maximum of K calls. Any call arriving when K calls are in progress is dropped.
(a) Let X_t denote the number of calls in progress at time t. Show that X is a Markov process, and draw a state transition diagram showing the transition rates.
Hint: Recall that the minimum of m iid exponential random variables with parameter μ is an exponential random variable with parameter $m\mu$.

(b) Under what conditions does an equilibrium distribution exist? Assuming these conditions, solve for the equilibrium distribution in terms of λ, μ, and K.
(c) Assuming that incoming calls see the equilibrium distribution, find the probability of call blocking. For $\lambda = 10$ and $\mu = 5$, how should we choose K such that the call blocking probability is less than 1%.
Remark: We are applying the celebrated PASTA (Poisson arrivals See Time Averages) property, which is heavily used in queueing theory.

Problem 8: Consider the setting of the previous problem, but now under the assumption that calls arriving when K calls are in progress are queued indefinitely and served in first come first served order when a telephone line becomes free. Thus, there is no call blocking in the system, but it is possible that the queue length becomes very large.
(a) Repeat parts (a) and (b) of the previous problem.
(b) Find the average number of calls (in progress, or in the queue) in the system. Obtain a numerical value for this average when $\lambda = 10$, $\mu = 5$, and $K = 10$.