ECE 235: Problem Set 6 (and recap of random processes through linear systems)

Assigned: Wednesday, November 25
Due: Thursday, December 3 (by noon, in course homework box)
Reading: Chapters 7 and 8 (you only need to go over the highlights covered in class); also prior material on Markov processes
Office hours for homework: For next week, I will hold office hours Wednesday December 2 (10-noon) instead of on Monday;
Final exam and review: The final exam is on Friday, December 11, 4-7 pm in the regular classroom; I will hold special office hours for the final exam on Friday, December 4, 9-11 am.

Random processes through linear systems

Suppose that a random process X is the input to a linear time-invariant (LTI) system with impulse response h. Let Y denote the random process at the output of the system. That is,

$$Y(t) = (X * h)(t) = \int_{-\infty}^{\infty} X(s)h(t-s)\, ds$$

Fact: If X is Gaussian, then X and Y are jointly Gaussian random processes.

Fact: If X is WSS, then X and Y are jointly WSS. The mean function of Y is a constant, given by

$$\mu_Y = \mu_X \int h(t)dt$$

The crosscorrelation function (and cross-spectrum) is given by

$$R_{YX}(\tau) = (R_X * h)(\tau)$$
$$S_{YX}(f) = S_X(f)H(f)$$

The autocorrelation function is given by

$$R_Y(\tau) = (R_{YX} * h_{mf})(\tau)$$
$$S_Y(f) = S_{YX}(f)H^*(f) = S_X(f)|H(f)|^2$$

where $h_{mf}(t) = h(-t)$.

Discrete Time Random Processes

Most of the standard results we know apply to both discrete and continuous time processes. However, we need some additional notation to talk about discrete time random processes through discrete time linear systems. Discrete time random processes are important because these are what we deal with when doing discrete time signal processing in communication transmitters and receivers. Moreover, while a communication system may involve continuous time signals, computer simulation of the system must inevitably be done in discrete time.

z-transform: The z-transform of a discrete time signal $s = \{s(n)\}$ is given by

$$S(z) = \sum_{n=-\infty}^{\infty} s(n)z^{-n}$$

The operator z^{-1} corresponds to a unit delay. We allow the variable z to take complex values. We are often most interested in $z = e^{j2\pi f}$ (on the unit circle), at which point the z-transform reduces to a discrete time Fourier transform (see below).

Remark: Given the z-transform of $S(z)$ expressed as a power series in z, you can read off $s(n)$ as the coefficient multiplying z^{-n}.

Discrete Time Fourier transform (DTFT): The DTFT of a discrete time signal s is its z-transform evaluated at $z = e^{j2\pi f}$; i.e., it is given by

$$S(e^{j2\pi f}) = S(z)|_{z=e^{j2\pi f}} = \sum_{n=-\infty}^{\infty} s(n)e^{-j2\pi fn}$$

It suffices to consider $f \in [0, 1]$, since $S(e^{j2\pi f})$ is periodic with period 1.

Convolution: If $s_3 = s_1 * s_2$ is the convolution of two discrete time signals, then $S_3(z) = S_1(z)S_2(z)$.

Discrete time random processes through discrete time linear systems

Let $X = \{X(n)\}$, a discrete time random process, be the input to a discrete time linear time invariant system with impulse response $h = \{h(n)\}$, and let $Y = \{Y(n)\}$ denote the system output.

As usual, let $h_{mf}(n) = h(-n)$ denote the impulse response for the matched filter for h.

Exercise: Show that $H_{mf}(z) = H(z^{-1})$ (assuming h is a real-valued impulse response). Infer that $H_{mf}(e^{j2\pi f}) = H^*(e^{j2\pi f})$.

Fact: If X is WSS, then X and Y are jointly WSS with

$$R_{YX}(k) = (R_X * h)(k)$$

$$R_Y(k) = (R_X * h * h_{mf})(k)$$

Power Spectral Density:

For a WSS discrete time random process X, the PSD is defined as the DTFT of the autocorrelation function. However, it is often convenient to also consider the z-transform of the autocorrelation function. As before, we use a unified notation for the z-transform and DTFT, and define the PSD as follows:

$$S_X(z) = \sum_{n=-\infty}^{\infty} R_X(n)z^{-n}$$
$$S_X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} R_X(n)e^{-j2\pi fn}$$

Similarly, for X, Y, jointly WSS, the cross-spectral density S_{XY} is defined as the z-transform or DTFT of the crosscorrelation function R_{XY}.

Fact: If X is WSS, and $Y = h * X$ (assume h and X real-valued), then X and Y are jointly WSS with

$$\begin{array}{ll} S_{YX}(z) = H(z)S_X(z) & S_{YX}(e^{j2\pi f}) = H(e^{j2\pi f})S_X(e^{j2\pi f}) \\ S_Y(z) = H(z)H(z^{-1})S_X(z) & S_Y(e^{j2\pi f}) = |H(e^{j2\pi f})|^2 S_X(e^{j2\pi f}) \end{array}$$

Problems

Practice problems (do not turn in): 7.10, 7.14, 7.18, 8.4, 8.6,

Problems 1-3: Problems 7.9, 7.13, 8.1

Problem 4: Let X denote a stationary Gaussian random process with zero mean and autocorrelation function $R_X(\tau) = e^{-|\tau|}$.
(a) **True or False:** X is a Markov process.
(b) Find the PSD $S_X(f)$.
(c) Find (numerically) the 99% power containment bandwidth B of X, which is defined by the equality

$$\int_{-B}^{B} S_X(f)df = 0.99 \int_{-\infty} \infty S_X(f)df$$

(d) Find the MMSE estimate of $X(t)$ given $X(0)$ and the error variance.
(e) Define $Y_t = X_t^2$ (i.e., Y is obtained by passing X through a squarer).

True or False: Y is a stationary random process.
True or False: Y is a Gaussian random process.
(f) Find the autocorrelation function of Y.
Hint: $\mathbb{E}[X^2(t)X^2(s)] = \mathbb{E}\left[X^2(t)\mathbb{E}[X^2(s)|X(t)]\right]$.

Problem 5: Let Z denote filtered WGN with PSD $S_Z(f) = I_{[-2,2]}(f)$.
(a) Find and sketch $R_Z(\tau)$.
Let $X_t = \frac{1}{10}\int_{t-10}^{t} Z_t dt$ denote a windowed average of Z.
(b) Is X WSS? Is X Gaussian?
(c) Find and plot the PSD of X.
Let Y denote the output when Z is passed through the RC-type lowpass filter $H(f) = \frac{1}{1+j2\pi f}$.
(d) Find the power of Y.

Problem 6: Consider a discrete time autoregressive, moving average (ARMA) process X defined by

$$X_n = \frac{2}{3}X_{n-1} + \frac{1}{2}X_{n-2} + U_n + U_{n-1}$$

where $\{U_k\}$ are i.i.d. $N(0,1)$ random variables.
(a) Find the PSD $S_X(z)$.
(b) Find the autocorrelation function $R_X(k,n) = \mathbb{E}[X_k X_n]$.
(c) Find and list some applications of ARMA models (e.g., a quick web search should suffice).
Remark: ARMA models are an important tool for modeling memory, and can be put in state space form: in the preceding example, the state would be $\mathbf{Z}(n) = (X_n, X_{n-1}, X_{n-2})^T$.