Partial Feedback Linearization "(PFL)" (Spong 94)

Linearize the non-linear EOM at each dt, and use a feedback law to control a partial subset of joints.

- For an UNDERACTUATED system, where

$$\boxed{m < n}$$

m: # actuated DOF ("degrees of freedom"); n: # DOF, total

- There remain $\boxed{\ell = n - m}$ "passive joints".

- <u>IF</u> we have a GOOD MODEL of the dynamics,
  - and if $m \geq \ell$ (more actuated than passive joints)
  - and if we can guarantee we can calculate any required matrix inverse or pseudo-inverse ("STRONG INERTIAL COUPLING"),

- <u>THEN</u>, → We can "shuffle around" the terms in the equations of motion to solve <u>DIRECTLY</u> for <u>ANY</u> m of n DOF's — not just the so-called "active" joints.

General Case:

- Nonlinear equations of motion (EOM) may be written as:

$$1.\quad \underbrace{M_{11}}_{(\ell \times \ell)} \underbrace{\ddot{q}_1}_{(\ell \times 1)} + \underbrace{M_{12}}_{(\ell \times m)} \underbrace{\ddot{q}_2}_{(m \times 1)} + \underbrace{h_1}_{(\ell \times 1)} + \underbrace{\phi_1}_{(\ell \times 1)} = \underbrace{0}_{(\ell \times 1)}$$

ℓ equations for ℓ "passive" joints

$$2.\quad \underbrace{M_{21}}_{(m \times \ell)} \ddot{q}_1 + \underbrace{M_{22}}_{(m \times m)} \ddot{q}_2 + \underbrace{h_2}_{(m \times 1)} + \underbrace{\phi_2}_{(m \times 1)} = \underbrace{T}_{(m \times 1)}$$

m equations for m "active" joints

- To control some set, m of the n DOF, of joints:
  - a) "Linearize" these non-linear EOM <u>at each "dt"</u> in time (as controller runs).
    - Plug in current q_i and $\dot{q}_i$ values, for all i.
  - b) Come up with a <u>control law</u> (e.g., simple "PD" proportional plus derivative control) that defines m of n <u>desired</u> accelerations: $\ddot{q}_i^{des}$ ← (for m DOF's) ← pick $\ddot{q}_i^{des}$ to get toward desired $q_i(t)$...
  - c) Use <u>matrix algebra</u> to <u>solve for</u> m elements in T ("torques") to achieve m values, $\boxed{\ddot{q}_i = \ddot{q}_i^{des}}$. ← "Shuffle" EOM, to solve for T!

a) Make all matrices have CONSTANTS at each dt. b) Move m joints toward m desired joint pos. c) Shuffle eqns to solve for T

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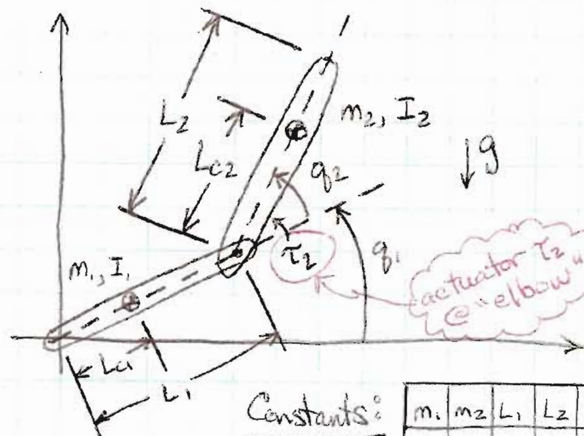
Acrobot Example:



Define some terms:

$$m_{11} = m_1 L_{c1}^2 + m_2\left(L_1^2 + L_{c2}^2 + 2L_1 L_{c2}\cos(q_2)\right) + I_1 + I_2$$
$$m_{22} = m_2 L_{c2}^2 + I_2$$
$$m_{12} = m_{21} = m_2\left(L_{c2}^2 + L_1 L_{c2}\cos(q_2)\right) + I_2$$
$$h_1 = -m_2 L_1 L_{c2}\sin(q_2)\dot{q}_2^2 - 2m_2 L_1 L_{c2}\sin(q_2)\dot{q}_2\dot{q}_1$$
$$h_2 = m_2 L_1 L_{c2}\sin(q_2)\dot{q}_1^2$$
$$\phi_1 = (m_1 L_{c1} + m_2 L_1) g\cos(q_1) + m_2 L_{c2} g\cos(q_1+q_2)$$
$$\phi_2 = m_2 L_{c2} g\cos(q_1+q_2)$$

Constants:

m_1	m_2	L_1	L_2	L_{c1}	L_{c2}	I_1	I_2	g
1	1	1	1	.5	.5	.2	1.0	9.8

Collocated: τ_2 set by choosing $\ddot{q}_2$. // **Non-collocated**: τ_2 set by choosing $\ddot{q}_1$.

collocated: plug in for $\ddot{q}_1$

non-collocated: plug in for $\ddot{q}_2$

$$1.\quad m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1 + \phi_1 = 0$$
$$2.\quad m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau_2$$

Collocated

A. Rewrite 1. w/ $\ddot{q}_1$ on lefthand side:

$$\boxed{\ddot{q}_1 = \frac{-1}{m_{11}}\left(m_{12}\ddot{q}_2 + h_1 + \phi_1\right)}$$

B. Plug this in for $\ddot{q}_1$ in 2.:

$$m_{21}\underbrace{\left[\frac{-1}{m_{11}}\left(m_{12}\ddot{q}_2 + h_1 + \phi_1\right)\right]}_{\ddot{q}_1} + m_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau_2$$

picking $\ddot{q}_2$ sets τ_2

C. Drive q_2 toward some desired q_2^{des}:

e.g.,
$$\boxed{q_2^{des} = \frac{2\alpha}{\pi}\tan^{-1}(\dot{q}_1)}$$

say, $\alpha = \frac{\pi}{2}$

D. Control laws (PD) sets $\ddot{q}_2^{des}$, to move q_2 toward q_2^{des}:

(Eq. 13)
$$\boxed{\ddot{q}_2 = v_2 = K_p(q_2^{des} - q_2) - K_D\dot{q}_2}$$

(Eq. 55)

E. Plug $\ddot{q}_2$ from D. into B. to get τ_2!!

Non-collocated

A. Rewrite 1. w/ $\ddot{q}_2$ on LHS:

$$\boxed{\ddot{q}_2 = \frac{-1}{m_{12}}\left(m_{11}\ddot{q}_1 + h_1 + \phi_1\right)}$$

B. Plug this in for $\ddot{q}_2$ in 2.:

$$m_{21}\ddot{q}_1 + m_{22}\left[\frac{-1}{m_{12}}\left(m_{11}\ddot{q}_1 + h_1 + \phi_1\right)\right] + h_2 + \phi_2 = \tau_2$$

picking $\ddot{q}_1$ sets τ_2

C. Drive q_1 toward some desired q_1^{des}:

$$\boxed{q_1^{des} = \frac{\pi}{2}}$$

1st link to be "upright" $\pi/2$

D. Control law (PD) picks a value for $\ddot{q}_1^{des}$, to drive q_1 toward q_1^{des}:

(Eq. 33) (Eq. 56)
$$\boxed{\ddot{q}_1 = v_1 = K_p(q_1^{des} - q_1) - K_D\dot{q}_1}$$

(p.2)

E. Plug $\ddot{q}_1$ from D. into B. $\rightarrow$ get τ_2!!

General Case ($m>1$, $\ell>1$ ... MATRIX equations!)

Both start with the same EOM in matrix form

Collocated | Non-collocated

$$1.\quad M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + h_1 + \phi_1 = 0$$

$$2.\quad M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + h_2 + \phi_2 = T$$

if collocated | if non-collocated

Collocated

plug in:

$$\ddot{q}_1 = -M_{11}^{-1}\left(M_{12}\ddot{q}_2 + h_1 + \phi_1\right)$$

A. Then 2. becomes:

$$-M_{21}\underbrace{M_{11}^{-1}\left(M_{12}\ddot{q}_2 + h_1 + \phi_1\right)}_{-\ddot{q}_1} + M_{22}\ddot{q}_2 + h_2 + \phi_2 = T$$

B. Collecting terms: (collocated case) (Eq. 11)

$$\boxed{\underbrace{\left(M_{22} - M_{21}M_{11}^{-1}M_{12}\right)}_{\bar{M}_{22}}\ddot{q}_2 + \underbrace{\left(h_2 - M_{21}M_{11}^{-1}h_1\right)}_{\bar{h}_2} + \underbrace{\left(\phi_2 - M_{21}M_{11}^{-1}\phi_1\right)}_{\bar{\phi}_2} = T}$$

(Eq. 13) $\boxed{\ddot{q}_2 = V_2}$

C. Write a control law to set V_2.

Non-collocated

plug in:

$$\ddot{q}_2 = -M_{12}^{\dagger}\left(M_{11}\ddot{q}_1 + h_1 + \phi_1\right)$$

where $M_{12}^{\dagger}$ is the PSEUDO-INVERSE of the $(\ell \times m)$ matrix M_{12}:

$$M_{12}^{\dagger} = M_{12}^{T}\left(M_{12}M_{12}^{T}\right)^{-1}$$

A. Then 2. becomes:

$$M_{21}\ddot{q}_1 - M_{22}\underbrace{M_{12}^{\dagger}\left(M_{11}\ddot{q}_1 + h_1 + \phi_1\right)}_{-\ddot{q}_2} + h_2 + \phi_2 = T$$

B. Collecting terms: (non-collocated case) (Eq. 35)

$$\boxed{\underbrace{\left(M_{21} - M_{22}M_{12}^{\dagger}M_{11}\right)}_{\tilde{M}_{21}}\ddot{q}_1 + \underbrace{\left(h_2 - M_{22}M_{12}^{\dagger}h_1\right)}_{\tilde{h}_2} + \underbrace{\left(\phi_2 - M_{22}M_{12}^{\dagger}\phi_1\right)}_{\tilde{\phi}_2} = T}$$

(Eq. 33) $\boxed{\ddot{q}_1 = V_1}$

C. Write a control law to set V_1.

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