Brief Overview (ECE 238) 3/29/2010

- Control
  "Robust"
  - good disturbance rejection (to perturbations)
  - low sensitivity (to changes in the plant)

- Digital Control
  issues include:
  - Sampling
    - time is discrete (sample period T)
    - signals are quantized (A/D & D/A resolutions)
  - Issues (w/ Sampling)
    - Aliasing
    - Digital filters
      - to emulate a CT plant
      - to filter signals
      - to implement control

- Control techniques
- System Identification (Sys ID)
  (- etc)

- Laplace vs. "Z" domain
  (discrete) transfer functions

\[ \mathcal{L} \]: Laplace transform operator
\[ \mathcal{Z} \]: (fancy-shaped "L")
\[ \mathcal{Z} \{ f(t) \} = SF(s) \] "S" operator corresponds to differentiation

\[ e.g. \]
\[ \begin{align*}
\mathcal{L} & \left( y + 2 \zeta w_n \dot{y} + w_n^2 y = K u \right) \\
& \left( s^2 + 2 \zeta w_n s + w_n^2 \right) Y(s) = K U(s)
\end{align*} \]

\[ g(s) : \frac{Y(s)}{U(s)} = \frac{K}{s^2 + 2 \zeta w_n s + w_n^2} \]

If signals are DISCRETE:
\[ Y(z) = \mathcal{Z}(y(k)) = \sum_{k=0}^{\infty} y_k z^{-k}, \quad r_0 < |z| < R_0 \]

(note: "CT" is for "continuous-time")
Discrete Transfer Functions (DTF's)

A DTF is the z-transform of the samples of the output when the input samples are the unit pulse @ $k=0$.

\[ 1 - \frac{1}{(1-e^{-Ts})} \]

G(s) responds to
- a unit step @ $t=0$, and
- a delayed step @ $t=T$.

In the Laplace domain, the response to a step pulse is:

\[
Y_1(s) = \frac{(1-e^{-Ts})G(s)}{s}
\]

The required DTF is the z-transform of the inverse of $Y_1(s)$:

\[
G(z) = Z\{Y_1(kT)\}
\]

\[
= Z\{L^{-1}\{Y_1(s)\}\} = Z\{Y_1(s)\}
\]

\[
= Z\{\frac{G(s)}{s}\} - Z\{e^{-Ts}\frac{G(s)}{s}\}
\]

"$e^{-Ts}$" is exactly a delay of one period, so:

\[
G(z) = (1-z)Z\left\{\frac{G(s)}{s}\right\}
\]
Example

Assume we have this plant

\[ G(s) = \frac{a}{s+a} \]

with ZOH at input to the plant... z-transform is??

From eqn at bottom of last page:

\[ G(z) = (1-z^{-1}) \mathcal{Z}\{ \frac{G(s)}{s+a} \} \]

\[ = (1-z^{-1}) \mathcal{Z}\{ \frac{1}{s} - \frac{a}{s+a} \} \]

Time function for this is:

\[ \mathcal{L}^{-1}\{ \frac{G(s)}{s} \} = 1(t) - e^{-at}1(t) \]

\[ @ t = kT \]

\[ = 1(kT) - e^{-akT}1(kT) \]

(begin aside...)

\[ X(z) = \mathcal{Z}\{ x(k) \} \]

\[ = \sum_{k=0}^{\infty} x_k z^{-k} , \text{for } |z| < R_e \]

For \( e^{-at}1(t) \), sampled at period "T", \( 1(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \)

So: \( x_k = e^{-akT}1(kT) \)

\[ \sum_{k=0}^{\infty} x_k z^{-k} = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^k \]

\[ \sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \]

\[ \frac{1}{p} \sum_{k=0}^{\infty} p^k = \frac{1}{p} \frac{1}{1-p} = \frac{1}{1-p} \sum_{k=0}^{\infty} p^k \]

Let \( p = e^{-aT}z^{-1} \)
So, using \[ \sum_{k=0}^{\infty} (e^{-at} z^{-1})^k = \frac{1}{1 - (e^{-at} z^{-1})} = \frac{z}{z - e^{-at}} \]

\[ Z\{ e^{-at} \} = \frac{Z}{Z - e^{-at}} \]

\[ \mathcal{L}^{-1}\left\{ \frac{G(s)}{S} \right\} = 1(kt) - e^{-akt} (kt) \]

\[ = e^{kt} - e^{-kt} \]

\[ = \left( \frac{Z}{Z - e^0} \right) - \left( \frac{Z}{Z - e^{-at}} \right) \]

\[ = \frac{Z(Z - e^{-at}) - Z(z^{-1})}{(z^{-1})(Z - e^{-at})} \]

\[ = \frac{Z e^{-at} - Z^2 + Z}{(z^{-1})(Z - e^{-at})} \]

\[ Z\left\{ \frac{G(s)}{S} \right\} = \frac{Z\left\{ \frac{G(s)}{S} \right\}}{Z\left\{ \frac{1}{S} \right\}} = \frac{1 - e^{-at}}{z - e^{-at}} \]

\[ 2\left( \frac{7}{3} \right) = 2\left( \frac{7}{3} \right) \]

\[ Z\{ G(s) \} = \frac{Z\{ G(s) \}}{Z\{ 1 \}} = \frac{1 - e^{-at}}{z - e^{-at}} \]

\[ \text{To try in MATLAB:} \]

\[ a = 50 \]
\[ G_s = tf([a], [1, a]) \]
\[ T = 0.001 \]
\[ G_z = c2d(G_s, T, 'matched') \]

Try also: Reehi, Justin, John
Filter design

- On previous pages, we wanted response of a CT plant, given ZOH at input...

- There are other options for filter design, too.

"Forward Euler"

- Here we estimate the derivative of sample k using values at k and k+1:

\[ \dot{x}_k \approx \frac{1}{T} (x_{k+1} - x_k) \]

So: since \( L[x^3] = Sx \),

this maps the "S" operator (differentiation) as:

\[ S \rightarrow \frac{1}{T} (z^1 - z^0) = \frac{1}{T} (z - 1) \]

Reverse mapping is:

\[ z \rightarrow (ST + 1) \]

- Another way to think about Forward Euler:

\[ x_{k+1} \approx x_k + T \dot{x}_k \] (numerical integration)

Note, this is a "conformal mapping" that maps the left-half plane (of the S-domain) to "z < 1", which includes regions outside the unit circle, i.e., Forward Euler can be UNSTABLE!
"Backward Euler"

Similar to forward Euler, but we now use $x_{k+1}$ in the relationship:

$$ x_{k+1} \approx \frac{1}{T} (x_{k+1} - x_k) $$

$$ (x_{k+1} = x_k + T \frac{dx_{k+1}}{dt} ) $$

<table>
<thead>
<tr>
<th>$S \rightarrow \frac{z-1}{zT}$</th>
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<tbody>
<tr>
<td>$Z \rightarrow \frac{1}{1 - sT}$</td>
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"Trapezoidal" aka "Tustin" aka "bilinear"

$$ \frac{1}{2} (\dot{x}_k + x_{k+1}) \approx \frac{1}{T} (x_{k+1} - x_k) $$

$$ x_{k+1} \approx x_k + \frac{T}{2} (\dot{x}_k + \dot{x}_{k+1}) $$

$$ \frac{1}{2} S(z_0 + z') \approx \frac{1}{2} (z' - z_0) $$

<table>
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<tr>
<th>$S \rightarrow \frac{2(z-1)}{T(z+1)}$</th>
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<tr>
<td>$Z \rightarrow \frac{2 + sT}{2 - sT}$</td>
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Let's model "$G(s) = \frac{b}{s+b}$" each way:

- **Forward Euler:** $\frac{b}{s+b} \rightarrow \frac{b}{(s+b)(z-1)+b}$
- **Backward Euler:** $\frac{b}{s+b} \rightarrow \frac{b}{s+b}$
- **Trapezoidal:** $\frac{bT^2 + bT}{(2+2bT)z + (bT-z)}$

Recall response of plant $G(s) = \frac{\alpha}{s+\alpha}$

@ input:

$$ \frac{1-e^{-bt}}{Z-e^{-bt}} $$