Brief Overview (ECE 238) 3/29/2010

^ (hopefully, "review"...)

- Control
  "Robust"
  - good disturbance rejection (to perturbations)
  - low sensitivity (to changes in the plant)

- Digital Control
  Issues include:
  - Sampling
    - time is discrete (sample period T)
    - signals are quantized (A/D & D/A resolutions)
    - Issues (w/ sampling)
      - Aliasing
      - Digital filters
        - to emulate a CT plant
        - to filter signals
        - to implement control
  - Control techniques
  - System Identification (sys ID)
  (- ... etc ...)

(note: "CT" is for "continuous time")

- Laplace vs. "Z" domain (discrete) transfer functions

$\mathcal{L}$: Laplace transform operator
(fancy-shaped "L")

$\mathcal{L}\{\dot{f}(t)\} = sF(s)$ ← "s" operator corresponds to differentiation

e.g.:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = Ku$$

$$\xrightarrow{\mathcal{L}} (s^2 + 2\zeta\omega_n s + \omega_n^2)Y(s) = KU(s)$$

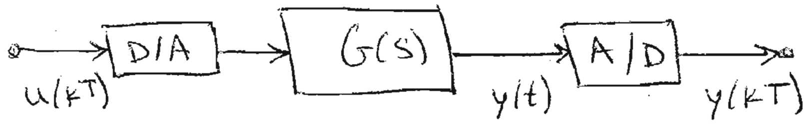
So: $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\xrightarrow{\mathcal{Z}}$ If signals are DISCRETE: $(y_0, y_1, y_2, \ldots, y_k)$

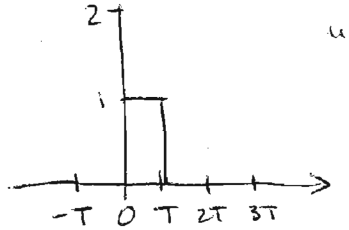
$$Y(z) \equiv \mathcal{Z}(y(k)) \equiv \sum_{k=-\infty}^{\infty} y_k z^{-k}, \quad r_0 < |z| < R_0$$

(in region of convergence)

Discrete Transfer Functions (DTF's)



A DTF is the z-transform of the samples of the output when the input samples are the <u>unit pulse @ k=0</u>.



"$1 - 1(t-T)$"

G(s) responds to a <u>unit step</u> @ $t=0$, and a <u>delayed step</u> @ $t=T$.

- in the Laplace domain, the response to a step pulse is:

$$Y_1(s) = (1-e^{-Ts})\frac{G(s)}{s}$$

- the required DTF is the z-transform of the samples of the <u>inverse</u> of $Y_1(s)$:

$$G(z) = \mathcal{Z}\{Y_1(kT)\}$$

$$= \mathcal{Z}\{\mathcal{L}^{-1}\{Y_1(s)\}\} \stackrel{\text{defined as}}{\equiv} \mathcal{Z}\{Y_1(s)\}$$

$$= \mathcal{Z}\left\{(1-e^{-Ts})\frac{G(s)}{s}\right\}$$

This has two parts

$$= \mathcal{Z}\left\{\frac{G(s)}{s}\right\} - \mathcal{Z}\left\{e^{-Ts}\frac{G(s)}{s}\right\}$$

"e^{-Ts}" is <u>exactly</u> a delay of one period, so:

$$\boxed{G(z) = (1-z^{-1})\,\mathcal{Z}\left\{\frac{G(s)}{s}\right\}}$$

Assume we have this plant

$\boxed{\text{Example}}$ " $G(s) = \dfrac{a}{s+a}$ " with ZOH at input to the plant... z-transform is??

from eqn at bottom of last page:

response to a step

partial fraction expansion...

$$G(z) = (1-z^{-1})\, \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= (1-z^{-1})\, \mathcal{Z}\left\{\frac{1}{s}\cdot\frac{a}{s+a}\right\}$$
$$= (1-z^{-1})\, \mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+a}\right\}$$

$$\frac{K_1}{s} + \frac{K_2}{s+a} = \frac{K_1(s+a) + K_2 s}{s(s+a)} = \frac{(K_1+K_2)s + K_1 a}{s(s+a)}$$
$\therefore K_1 = 1,\ K_2 = -K_1 = -1$

Time function for this is:

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(t) - e^{-at}1(t)$$

@ $t = kT$

$$= 1(kT) - e^{-akT}1(kT)$$

By definition for signal $\{x_0, x_1, \ldots x_k\}$ $X(z) \equiv \mathcal{Z}\{x(k)\} = \sum_{k=-\infty}^{\infty} x_k z^{-k}$

(begin aside...)

$$X(z) \equiv \mathcal{Z}\{x(k)\}$$
$$= \sum_{k=-\infty}^{\infty} x_k z^{-k}, \quad r_0 < |z| < R_0$$

---

for $e^{-at}1(t)$, sampled at period "T", $1(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$

So: $x_k = e^{-akT}1(kT)$

$\boxed{\text{infinite series}}$

$$\sum_{k=-\infty}^{\infty} x_k z^{-k} = \sum_{k=0}^{\infty} e^{-akT} z^{-k} = \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^k$$

$$\sum_{k=0}^{\infty} p^k = 1 + p + p^2 + \ldots + p^{\infty}$$

for $p<1$

$$\frac{1}{p}\sum_{k=0}^{\infty} p^k = \frac{1+p+p^2+\ldots+p^{\infty}}{p} = \frac{1}{p} + \sum_{k=0}^{\infty} p^k$$

$$\therefore \left(\tfrac{1}{p}\right)(1-p)\sum_{k=0}^{\infty} p^k = \left(\tfrac{1}{p}\right)1$$

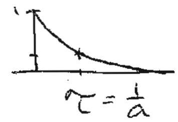
$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$$

Let $p = (e^{-aT}z^{-1})$

So, using $\sum_{k=0}^{\infty} (e^{-aT} z^{-1})^k = \dfrac{1}{1-(e^{-aT}z^{-1})} = \boxed{\dfrac{z}{z-e^{-aT}}}$

$$\therefore \boxed{\mathcal{Z}(e^{-akT}) = \frac{z}{z-e^{-aT}}}$$

$$\mathcal{L}(e^{-akT}) = \frac{1}{s+a}$$



(...end aside.)

(back to "$G(s) = \frac{a}{s+a}$" ...)

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(kT) - e^{-akT}(kT)$$

$$= e^{0kT} - e^{-akT}$$

$$= \left(\frac{z}{z-e^{0}}\right) - \left(\frac{z}{z-e^{-aT}}\right)$$

$$= \frac{z(z-e^{-aT}) - z(z-1)}{(z-1)(z-e^{-aT})}$$

$$= \frac{\cancel{z^2} - ze^{-aT} \cancel{- z^2} + z}{(z-1)(z-e^{-aT})}$$

$\mathcal{Z}\left(\frac{1}{s}\right) = \mathcal{Z}\left(\frac{1}{s+0}\right) = \frac{z}{z-1}$

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

for $G(s) = \frac{a}{s+a}$

$$\boxed{\mathcal{Z}\{G(s)\} = \frac{\mathcal{Z}\left\{\frac{G(s)}{s}\right\}}{\mathcal{Z}\left\{\frac{1}{s}\right\}} = \frac{(1-e^{-aT})}{(z-e^{-aT})}}$$

*** To try in MATLAB:**

```
a=50
Gs=tf([a],[1, a])
T = .001
Gz=c2d(Gs, T, 'matched')
```

Try also: 'zoh' 'tustin' 'foh'

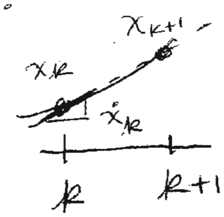
Filter design

- On previous pages, we wanted response of a CT plant, given ZOH @ input...

- There are <u>other</u> options for filter design, too.

"Forward Euler"

- Here we estimate the derivative of sample k using values at k & $k+1$:



$$\dot{x}_k \approx \frac{1}{T}\left(x_{k+1} - x_k\right)$$

So: since $\mathcal{L}\{\dot{x}\} = sX$,

this maps the "s" operator (differentiation) as:

$$\boxed{s \rightarrow \frac{1}{T}\left(z^1 - z^0\right) = \frac{1}{T}(z-1)}$$

reverse mapping is →

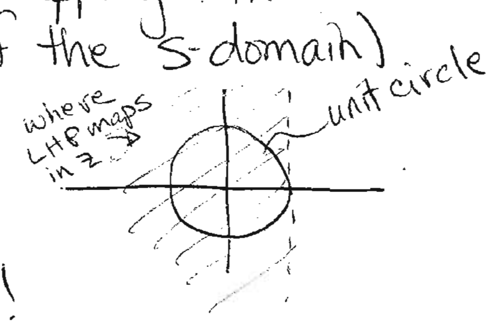
$$\boxed{z \rightarrow (sT+1)}$$

- Another way to think about forward Euler:

$$x_{k+1} \approx x_k + T\dot{x}_k \quad \text{(numerical integration...)}$$

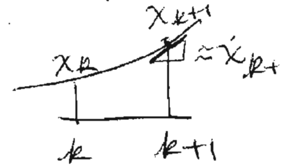
(just rewriting same relationship)

<u>Note</u>, this is a "conformal mapping" that maps the left half plane (of the s-domain) to "$z < 1$", which includes regions <u>outside</u> the unit circle! i.e., Forward Euler <u>can</u> be UNSTABLE!



"Backward Euler"

- Similar to forward Euler, but we now use $\dot{x}_{k+1}$ in the relationship:



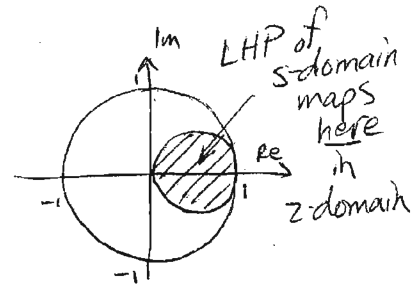
$$\dot{x}_{k+1} \approx \frac{1}{T}(x_{k+1} - x_k)$$

or

$$\left(x_{k+1} \approx x_k + T\dot{x}_{k+1} \right)$$

$$\boxed{S \to \frac{z-1}{zT}}$$

$$\boxed{z \to \frac{1}{1-sT}}$$



"Trapezoidal" aka "Tustin" aka "bilinear"

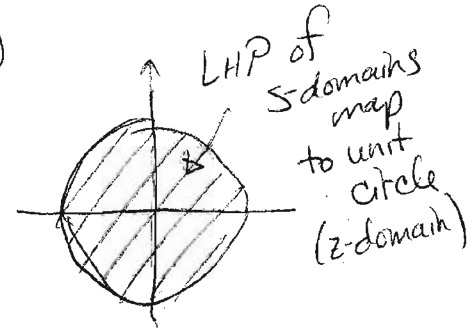
$$\frac{1}{2}(\dot{x}_k + \dot{x}_{k+1}) \cong \frac{1}{T}(x_{k+1} - x_k)$$

$$x_{k+1} \approx x_k + \frac{T}{2}(\dot{x}_k + \dot{x}_{k+1})$$

$$\frac{1}{2}s(z^0 + z^1) \approx \frac{1}{T}(z^1 - z^0)$$

$$\boxed{S \to \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$\boxed{z \to \frac{2+sT}{2-sT}}$$



Let's model "$G(s) = \frac{b}{s+b}$" each way:

F. Euler: $\frac{b}{s+b} \to \frac{b}{(\frac{1}{T})(z-1)+b} \to \left(\frac{bT}{z-(1-bT)}\right)$

B. Euler: $\frac{b}{s+b}$ - - - - - - $\to \left(\frac{bTz}{bTz-1}\right)$

Trap.: $\left(\frac{bTz + bT}{(2+bT)z + (bT-2)}\right)$

Recall response of Plant, $G(s) = \frac{a}{s+a}$, w/ ZOH @ input:

$$\frac{1-e^{-bT}}{z-e^{-bT}}$$