Magnetic Bearing Lab # 1: Analytical Modeling of a Magnetic Bearing System*

Nancy Morse, Roy Smith and Brad Paden
e-mail: nancy@me-server.ucsb.edu

May 29, 1996

Control system design is a multistage process involving much more than the design of the controller itself. Before a controller can be designed, the designer must have sufficient knowledge of the system to be controlled. The designer begins by collecting information about the system from all available sources and then representing this information in the form of a system model. One source of information is the physics governing the system. An analysis of these physical laws will yield differential or difference equations which describe the motion of the system in response to certain input signals.

Experimental input/output data taken on the actual system is another common source of system information. A given input signal is used to drive the system and the system response to this signal is recorded. This experimental input/output relationship then processed to determine a transfer function model of the system.

Once a suitable system model is obtained, the controller design can begin. Typically a first design will be evaluated in computer simulation using a software package such as MATLAB or Simulink. Performance and stability of the controlled system are evaluated and design iterations of the controller are performed until a controller is found which meets the design requirements. Next, the controller is built and its performance is evaluated on the actual system.

You will notice that the modeling process we have described above is composed of two parts: the derivation of a physically based analytical model and the use of experimental data to determine a model. Both phases are important for determining an appropriate model.

*The following software is required to perform this lab: MATLAB and μ-TOOLS.

If the models obtained from each process differ greatly, then more work must be done to determine the source of the difference. The differences may be the caused by an error or over simplification in the physical analysis of the system, or they may be the result of experimental data which is insufficient to capture essential system characteristics. Understanding the source of the differences will yield additional information about the system and can lead to a more accurate system model.

1 Analytical Model Derivation

In this lab, we will focus on the derivation of a physically based model for the magnetic bearing system. The modeling will be divided into two parts. In Section 1.1, we will view the system rotor as a rigid body and derive the corresponding equations of motion. Then in Section 1.2, we will include the flexible motion of the rotor. Section 1.2 is optional and is intended for a more advanced course in control system design. We will use MATLAB to compile the models and to determine certain model characteristics.

The system we wish to control is the Magnetic Moments MBC 500 magnetic bearing system. A diagram of this system is shown in Figure 1. This system contains a stainless steel shaft or rotor which can be levitated using eight "horseshoe" electromagnets, four at each end of the rotor. Hall effect sensors placed just outside of the electromagnets at each end of the rotor measure the rotor end displacement. This system is a four degree of freedom system with two degrees of freedom at each end of the rotor. These two degrees of freedom are translation in the horizontal direction perpendicular to the z axis (x_1 and x_2) and translation in the vertical direction (y_1 and y_2). Also included in the MBC 500 package are four controllers which levitate the bearing when connected in feedback. On the front panel are four switches for disconnecting each of the controllers so that any one or all of them can be replaced by an external controller.

1.1 Rigid Body Rotor Analysis and System Modeling

For our first and simplest analysis of the system, we assume that the rotor acts as a rigid body. A rigid body is one which does not change shape. Therefore, we assume that the rotor does not bend but rather experiences only translational or rotational motion. In addition, we

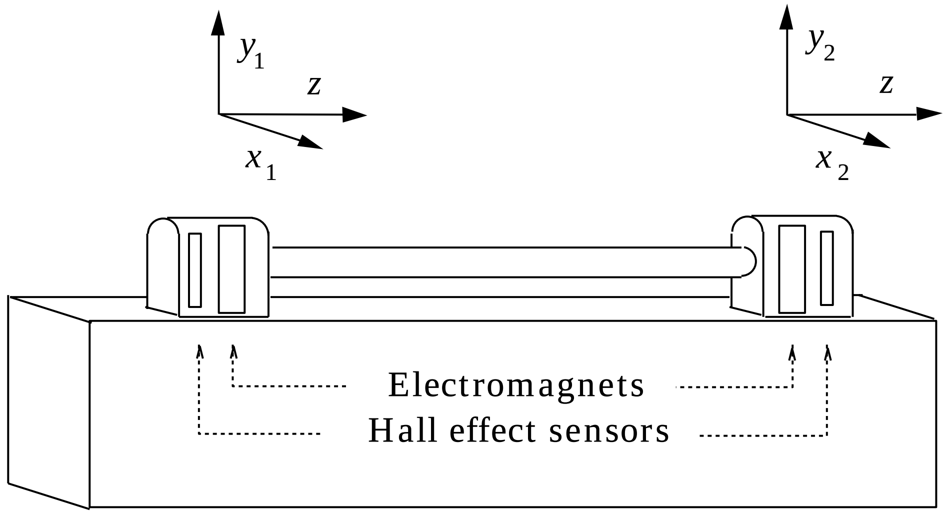

Figure 1: MBC 500 System Configuration

assume that the horizontal and vertical dynamics, i.e. the x and y directions, are uncoupled. Coupling of these dynamics would occur if the rotor were spinning or if the actuators or sensors were misaligned; however, we will assume that these effects are small. Our system, in theory, operates identically in the x and y directions except for the additional constant force due to gravity acting in the y direction. This constant force is not linear and cannot be modeled by a linear system model. Therefore, we must neglect gravity in the linear y direction analysis. Thus, analysis of the x and y directions is identical, and we will focus our analysis strictly on the horizontal or x direction motion.

For the following analysis, the system configuration and parameters defined in Figure 2 and Table 1 and Table 2 apply.

The nominal or desired rotor position corresponds to $x_1 = 0$ and $x_2 = 0$ (or equivalently $X_1 = 0$ and $X_2 = 0$ or $x_0 = 0$ and $\theta = 0$). In this position, the rotor is centered horizontally with respect to the front and back electromagnets on each end, and its long axis is parallel to the z axis.

An analysis of the geometry of the rotor will yield the following relationships.

$$
\begin{aligned}
x_1 &= x_0 - (\frac{L}{2} - l)\sin\theta \\
x_2 &= x_0 + (\frac{L}{2} - l)\sin\theta \\
X_1 &= x_0 - (\frac{L}{2} - l_2)\sin\theta
\end{aligned}
$$

Symbol	*Description*
x_0	The horizontal displacement of the center of mass of the rotor.
x_1 and x_2	The horizontal displacements of the rotor at left and right bearing positions, respectively
X_1 and X_2	The horizontal displacements of the rotor at left and right Hall-effect sensor positions, respectively
θ	The angle that the long axis of the rotor makes with the z axis
F_1 and F_2	The forces exerted on the rotor by left and right bearings, respectively

Table 1: System Variables

Symbol	*Description*	*Value*
L	Total length of the rotor	0.269m
l	Distance from each bearing to the end of the rotor	0.024m
l_2	Distance from each Hall-effect sensor to the end of the rotor	0.0028m
I_0	Moment of inertia of the rotor with respect to rotation about an axis in the y direction $I_0 = \frac{1}{12}mL^2$	1.5884×10^{-3}kg m^2
m	Mass of the rotor	0.2629kg

Table 2: System Parameters

$$X_2 = x_0 + (\frac{L}{2} - l_2)\sin\theta$$

Considering the physical limits of our system, we may assume that θ is small. This allows us to make the first order approximations: $\sin\theta \cong \theta$ and $\cos\theta \cong 1$.

We now review Newton's laws and how they can be used to find the equations of motion for simple rigid body mechanical systems. For the rotor analysis, we will be using force and moment balance to help derive the system dynamics. First, we have the force balance equation:

$$\sum \vec{F} = m\vec{a}.$$

In this equation, $\sum \vec{F}$ is the summation of all external forces applied to the system, m is the rotor mass, and $\vec{a}$ is the acceleration of the center of gravity of the system. Next, from

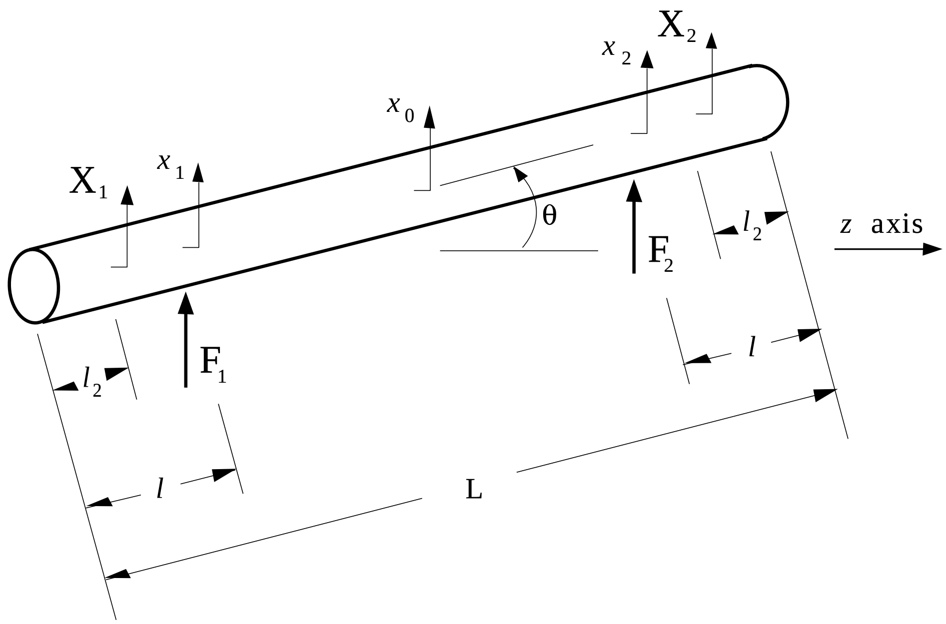

Figure 2: Rotor Configuration

moment balance we have the relationship

$$\sum \vec{M} = I\vec{\alpha}.$$

Here $\sum \vec{M}$ is the summation of all moments applied externally to the system, I is the rotational moment of inertia of the system about the axis through the center of gravity and in the direction of rotation, and $\vec{\alpha}$ is the angular acceleration of the system. In general, moments and forces are interrelated in the following way:

$$\vec{M} = \vec{r} \times \vec{F}.$$

This relationship is shown pictorially in Figure 3(a). In this relationship, $\vec{r}$ is any vector pointing from the point 0 to the line of application of the force $\vec{F}$. If the vector $\vec{r}$ is chosen to be perpendicular to the line of application of the force $\vec{F}$ as shown in Figure 3(b), then the above equation reduces to

$$M = rF.$$

For the examples in Figure 3, the sense of the moment is counter-clockwise.

We will now use force and moment balance to derive the non-linear differential equations governing the rigid body motion for the bearing rotor shown in Figure 2 above. We assume

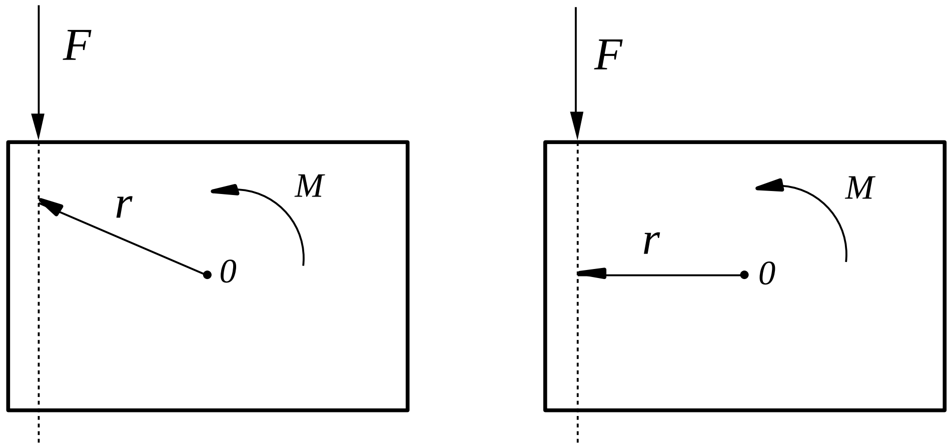

Figure 3: Force/Moment Relationships

motion in only one plane (i.e. in the x direction only). The equations of motion are:

$$\begin{aligned} \Sigma F &= m\ddot{x}_0 = F_1 + F_2 \\ \Sigma M &= I_0\ddot{\theta} = F_2(\frac{L}{2} - l)\cos\theta - F_1(\frac{L}{2} - l)\cos\theta \end{aligned}$$

Exercise 1: Use the small angle θ assumption and linearize the equations of motion obtained above.

Exercise 2: A "state-space" system description takes the form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where x is the state vector, u is the system input vector and y is the system output vector. With F_1 and F_2 as input variables and X_1 and X_2 as output variables, show that the equations of motion obtained in Exercise 2 can be expressed in state-space form as the following multiple-input/multiple-output (MIMO) system.

$$\begin{bmatrix} \dot{x}_0 \\ \ddot{x}_0 \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m} & \frac{1}{m} \\ 0 & 0 \\ -\frac{1}{I_o}\left(\frac{L}{2} - l\right) & \frac{1}{I_o}\left(\frac{L}{2} - l\right) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\left(\frac{l}{2} - l_2\right) & 0 \\ 1 & 0 & \left(\frac{l}{2} - l_2\right) & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \theta \\ \dot{\theta} \end{bmatrix}$$

What is the state vector for this system? What are the eigenvalues of the A matrix corresponding to this system?

Recall that the first order approximation of a continuously differentiable function $f(x)$ at a point $x = a$ is given by:

$$f(x) \cong f(a) + f'(a)(x - a).$$

Similarly, for a function of two variables, a first order approximation at a point $(x, y) = (a, b)$ is given by:

$$f(x, y) \cong f(a, b) + \left[\frac{\partial f}{\partial x}(a, b)\right](x - a) + \left[\frac{\partial f}{\partial y}(a, b)\right](y - b).$$

Now consider the non-linear relationship which was given in the bearing manual describing the bearing force on the rotor as a function of rotor displacement and control current. Given that $i_{control_i}$ is the control current into bearing i in amps, and x_i is measured in meters, we can express the i^{th} force as

$$F_i = k\frac{(i_{control_i} + 0.5)^2}{(x_i - 0.0004)^2} - k\frac{(i_{control_i} - 0.5)^2}{(x_i + 0.0004)^2}$$

where $k = 2.8 \times 10^{-7}\text{Nm}^2/\text{Amp}^2$. Notice that $F_i = 0$ when both $x_i = 0$ and $i_{control_i} = 0$. Thus, the point $(x_i, i_{control_i}) = (0, 0)$ is an equilibrium point for the function F_i. This bearing non-linearity enters the system as an input or actuator non-linearity.

Exercise 3: Using a linear Taylor series approximation, linearize the bearing non-linearity with respect to its two variables x_i and $i_{control_i}$ about the equilibrium point $(x_i, i_{control_i}) = (0, 0)$.

Exercise 4: Now combine the results of Exercise 3 with the expressions for x_1 and x_2 given in the beginning of this section to express F_1 and F_2 as a linear function of x_0 and θ and of $i_{control_1}$ and $i_{control_2}$, respectively.

Exercise 5: Substitute the expressions for F_1 and F_2 which you obtained in Exercise 4 into the rigid body state space description of the system given in Exercise 2 to obtain a new state space system model which includes the actuator dynamics. Your new model should have $i_{control_1}$ and $i_{control_2}$ as input variables and should take the following form:

$$\begin{bmatrix} \dot{x}_0 \\ \ddot{x}_0 \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = A \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \theta \\ \dot{\theta} \end{bmatrix} + B \begin{bmatrix} i_{control_1} \\ i_{control_2} \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = C \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \theta \\ \dot{\theta} \end{bmatrix}$$

where A, B, and C are your new state space matrices.

Exercise 6: For the state space model obtained in Exercise 5, what are the eigenvalues of the new system A matrix? Is this system stable or unstable? What are the natural frequencies of this system?

Now consider the sensor non-linearity. The bearing manual gives the voltage sensed by the controller, V_{sense_i}, as a function of rotor displacement, X_i. The relationship is as follows:

$$V_{sense_i} = 5000X_i + (25 \times 10^9)X_i^3$$

where X_i is measured in meters.

Exercise 7: Find a linear approximation to the sensor non-linearity at $X_i = 0$ and rewrite the system model with V_{sense_1} and V_{sense_2} as the output variables. Does this change effect the system eigenvalues?

Finally, there is a current amplifier at the input of our system that regulates the current into the bearings. The magnetic bearing manual tells us that this amplifier has the following dynamical characteristic:

$$\frac{d}{dt}\left(i_{control_i}\right) = -\frac{1}{2.2 \times 10^{-4}} i_{control_i} + \frac{0.25}{2.2 \times 10^{-4}} V_{control_i}.$$

The amplifier input is $V_{control_i}$ and its output is $i_{control_i}$. $V_{control_1}$ and $V_{control_2}$ are the voltages that the controller produces to control the bearing.

Exercise 8: What is the range of frequencies passed by the filtering action of the current amplifier? Is this a low pass, high pass or band pass filter?

Exercise 9: Add the current amplifier dynamics to the rigid body model obtained in Exercise 7. Let $V_{control_1}$ and $V_{control_2}$ be your input variables, V_{sense_1} and V_{sense_2} be your output variables, and choose

$$\begin{bmatrix} x_0 \\ \dot{x}_0 \\ \theta \\ \dot{\theta} \end{bmatrix}$$

as your state vector.

Exercise 10: Find the eigenvalues of the A matrix of the model obtained in Exercise 9. Note which eigenvalues correspond to rigid body rotor modes and which correspond to current amplifier dynamics.

The magnetic bearing manual gives the following nominal transfer function for each of the on-board controllers:

$$V_{control_i} = \frac{1.41(1 + 8.9 \times 10^{-4} s)}{(1 + 3.3 \times 10^{-4} s)(1 + 2.2 \times 10^{-5} s)} V_{sense_i} = C(s)\ V_{sense_i}.$$

For our controller design, we will attempt to replace just the controller $C_{x_1}(s)$ which maps V_{sense_1} to $V_{control_1}$. Thus, the system seen by our controller will be as shown in Figure 4. However, because of our simplifying assumption that the x and y rotor motions are decoupled, an equivalent system configuration is as shown in Figure 5. In order to obtain an analytical model for the system to be controlled, we must include the controllers $C_{x_2}(s)$, $C_{y_1}(s)$, and $C_{y_2}(s)$ in feedback as shown in Figure 5.

Exercise 11: Execute the following commands in MATLAB to convert the controller transfer function $C(s)$ to a state space model with system matrices `acont`, `bcont`, `ccont`, and `dcont`.

```
num = [1.25e-03, 1.41];
den = [7.2600e-09, 3.5200e-04, 1];
[acont, bcont, ccont, dcont] = tf2ss(num, den);
```

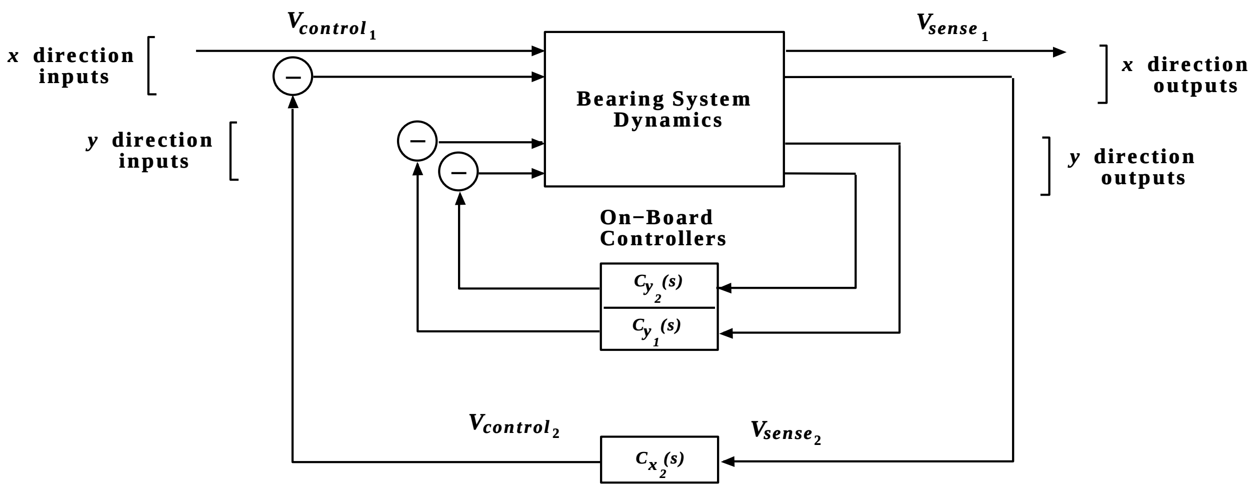

Figure 4: Bearing System Seen by Controller C_{x_1} — x and y Directions Coupled

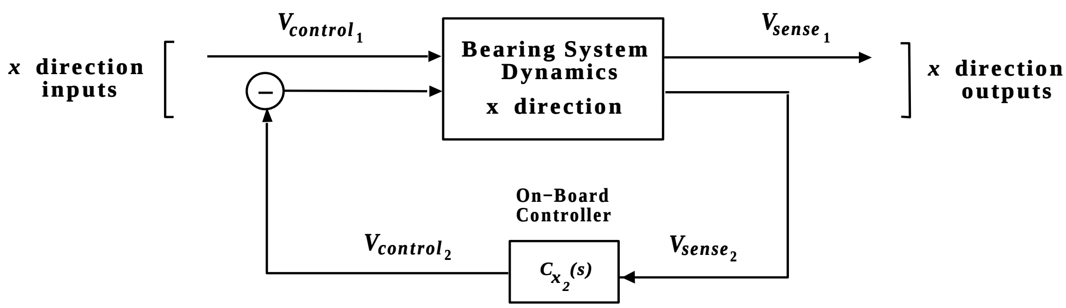

Figure 5: Bearing System Seen by Controller C_{x_1} — x and y Directions Decoupled

Now, using the following MATLAB and μ-TOOLS commands, add the controller dynamics in feedback around the bearing system model as shown in Figure 5. In the following commands, we assume the system model derived in Example 9 has `a`, `b`, `c`, and `d` as its system matrices. The following algorithm creates system matrices `abar`, `bbar`, `cbar`, and `dbar` for the system of Figure 5.

```
cont = pck(acont, bcont, ccont, dcont);
sys = pck(a, b, c, d);
sysbar = starp(sys, cont);
[abar, bbar, cbar, dbar] = unpck(sysbar);
```

Exercise 12: Assuming that the system model of Exercise 11 is defined in MATLAB with `abar`, `bbar`, `cbar`, and `dbar` as its system matrices, use the following MATLAB commands to plot the Bode diagram of the transfer function from input $V_{control_1}$ to output V_{sense_1}.

$$\texttt{freq} = \texttt{logspace}(1, 4, 200);$$

$$\texttt{bode}(\texttt{abar}, \texttt{bbar}, \texttt{cbar}, \texttt{dbar}, 1, \texttt{freq});$$

The system model obtained from Exercise 11 is the rigid body analytical model for the system we wish to control and will be referred to as the "rigid body model" hereafter.

1.2 Flexible Rotor Analysis and System Modeling

In the above analysis, we have assumed that the rotor is a rigid body. However, during levitation, the actual rotor does experience bending. There are an infinite number of bending modes excited in the rotor, however we will only attempt to model the two modes of lowest frequency. The bending shapes of these modes are described by Figure 6. If we assume that

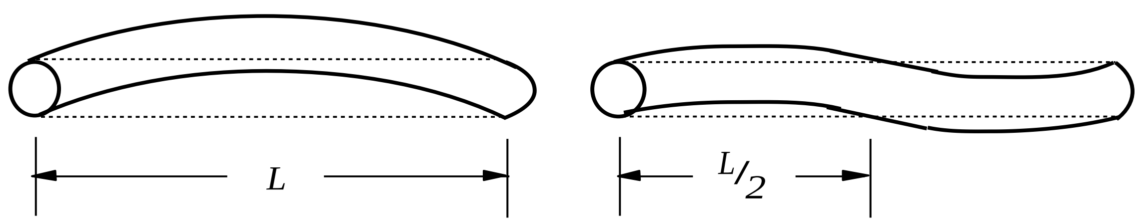

Figure 6: Bending Modes

no rigid body motion is excited in the system, we can obtain equations of motion describing the system motion due strictly to the rotor bending [1]. We first define a vector of amplitude variables

$$a \triangleq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

and relate it to the rotor displacement as follows:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -1.93745e+00 & -1.83546e+00 \\ -1.93745e+00 & 1.83546e+00 \end{bmatrix} a$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.19029e+00 & -6.08354e-01 \\ -1.19029e+00 & 6.08354e-01 \end{bmatrix} a.$$

We also define a mass matrix

$$M = \begin{bmatrix} 2.72146e-01 & 0 \\ 0 & 2.62297e-01 \end{bmatrix},$$

a stiffness matrix

$$K = \begin{bmatrix} 6.05762e+06 & 0 \\ 0 & 4.43630e+07 \end{bmatrix},$$

and a vector forcing function

$$P = \begin{bmatrix} -1.19029e+00 & -1.19029e+00 \\ -6.08354e-01 & 6.08354e-01 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

The following differential equation describes the bending motion of the rotor:

$$M\ddot{a} + Ka = P.$$

Notice the similarity between the differential equation given above and the standard differential equation of motion for a spring/mass system. Thus, in bending, the rotor behaves much like an interconnection of springs and masses.

Exercise 13: From the differential equation above, define an appropriate state vector and rewrite the rotor bending equations in state space form. As in the rigid body case, choose F_1 and F_2 as your input variables and X_1 and X_2 as your output variables. What are the frequencies of oscillation of the flexible system modes as described in the equations of motion above?

We have derived the equations of motion for the rotor in rigid body motion alone and in bending motion alone. Therefore, we may sum both of these responses to obtain the total displacement of the rotor. (See for example [1].)

Exercise 14: Express the rotor displacement as a sum of the displacements due to rigid body motion and bending motion. Use this to obtain a combined model which describes the bearing system response in terms of both rigid and flexible system modes. Let F_1 and F_2 be your input variables and X_1 and X_2 be your output variables.

Exercise 15: Now use the linear approximations to the input and output non-linearities that you obtained in Exercises 4 and 7 to add the actuator and sensor dynamics to the combined model derived in Exercise 14. Your input variables should be $i_{control_1}$ and $i_{control_2}$ and your output variables should be V_{sense_1} and V_{sense_2}. Be sure to use the total displacement (the displacement due to both rigid and bending motion) in the expression for the bearing forces F_1 and F_2.

Exercise 16: Add the current amplifier dynamics to the combined model obtained in Exercise 15. Let $V_{control_1}$ and $V_{control_2}$ be your input variables and V_{sense_1} and V_{sense_2} be your output variables.

Exercise 17: Find the eigenvalues of the A matrix of the system model of Exercise 16. Relate each of them to the rigid body dynamics, bending dynamics or the current amplifier.

Exercise 18: Using the MATLAB algorithms of Exercise 11, add the on-board controller in feedback around the system model obtained in Exercise 16. We will refer to this model as the "total system model" hereafter.

Exercise 18: Using the MATLAB commands given in Exercise 12, plot the Bode diagram of the total system model. Compare this Bode diagram to the diagram you obtained for the rigid body model in Exercise 12. What differences do you see and how do you account for these? At what frequencies do the primary differences occur?

Exercise 19: Using the Bode diagram obtained in Exercise 18, find the bandwidth of the total system model. Is it significantly different than the bandwidth of the rigid body model?

References

[1] Craig, Roy R., Jr., **Structural Dynamics**, John Wiley & Sons, Inc., 1981, pp. 210-216 and pp. 251-257.

Version	*Date*	*Modifications*
2	4/30/96	Lab software requirements list added on p. 1
		The phrase "the rotor is centered vertically" is changed to "the rotor is centered horizontally" on p. 4
		Lables modified in Figure 2 on p. 5
		Controller transfer function changed on p. 9
		Equations for `num` and `den` changed on p. 9
		Flexible system matrices changed on pp. 11,12
		Minor corrections made throughout text