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Hybrid Control for the Pendubot

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Abstract: A new hybrid controller has been applied for feedback stabilization of a Pendubot, whose dynamics show second order nonholonomic properties. Experimental results for different test cases show that hybrid control outperforms the existing control algorithm. This successful implementation provides a novel alternative for the control of under-actuated mechanical systems. Many such systems are subject to second order nonintegrable differential constraints.

1 Introduction

A PENDUBOT [1] is a two-link (two-degree-of-freedom) planar robot, whose first link (shoulder) is actuated and second link (elbow) is not actuated. It is an under-actuated mechanical system(see Figure 1).

Under-actuated mechanical systems generally have multiple equilibria. For the Pendubot, the position shown in Figure 1 is an unstable inverted equilibrium, which is the most difficult case for feedback stabilization among all the equilibria. The challenge for control of under-actuated mechanical systems is to use actuated links to control the passive links by utilizing the nonlinear dynamic coupling between them.

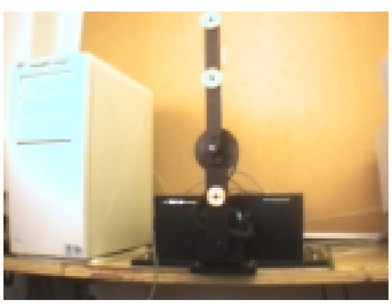

Figure 1: Picture of the Pendubot

The Pendubot possesses some unique features and challenges for control research not found in other under-actuated mechanical systems, such as control of a cart [2] and control of an Acrobot [3]. The Pendubot is simple enough to permit complete dynamic analysis and experimentation, but complex enough for investigating many advanced nonlinear control approaches, such as optimal control, robust and adaptive control, intelligent control, and hybrid and switching control.

Specifically, the Pendubot exhibits second order nonholonomic properties, which means the dynamics of the Pendubot are subject to second order nonintegrable differential constraints. Control of second order nonholonomic systems has been one of the most active research areas in the last few years. The difficulty is that for a large class of non-holonomic systems, it is impossible to use smooth feedback to stabilize the system around an equilibrium even locally. Hybrid control has been considered as a good choice.

As reported in [4], many theoretical studies have been performed for control of nonholonomic systems. However, few results have been implemented. There is a strong need for the experimental study of control of nonholonomic systems. A novel hybrid control approach for feedback stabilization of second order nonholonomic systems is presented in [5]. In this paper, the application of the hybrid control approach for feedback stabilization of a Pendubot is presented.

2 Dynamic Model and Control Properties of the Pendubot

2.1 Dynamic Model of the Pendubot

The dynamic model of the Pendubot can be easily derived by using Euler-Lagrange equations [6]. Through reparameterization, the dynamical parameters of the Pendubot can be grouped into five parameters $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. Then, we have the following dynamic system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + e(q) = \tau \qquad (1)$$

$$M(q) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 cos(q_2) & \theta_2 + \theta_3 cos(q_2) \\ \theta_2 + \theta_3 cos(q_2) & \theta_2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} \theta_6 - \theta_3 sin(q_2)\dot{q}_2 & -\theta_3 sin(q_2)(\dot{q}_2 + \dot{q}_1) \\ \theta_3 sin(q_2)\dot{q}_1 & 0 \end{bmatrix}$$

$$
\begin{aligned}
e(q) &= \begin{bmatrix} \theta_4 gcos(q_1) + \theta_5 gcos(q_1+q_2) \\ \theta_5 gcos(q_1+q_2) \end{bmatrix} \\
\tau &= \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

where $q = [q_1, q_2]^t$ is the vector of generalized coordinates (joint variables for robotic manipulator). Here, q_1 represents the control link, and q_2 represents the under-actuated link (see Figure 2). $M(q)$ is the 2×2 inertia matrix, $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centripetal torques, $e(q)$ is the gravitational term, and τ is the vector of control torque. The dynamic parameters of the Pendubot are identified as follows [1]:

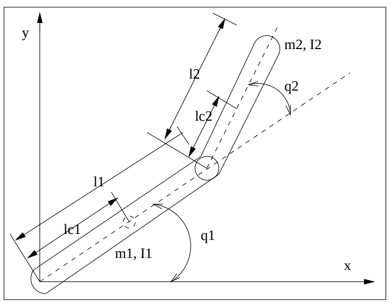

Figure 2: Dynamics of the Pendubot

$$
\begin{aligned}
\theta_1 &= 0.0308vs^2, \quad \theta_2 = 0.0106vs^2 \\
\theta_3 &= 0.0095vs^2, \quad \theta_4 = 0.2086vs^2/m \\
\theta_5 &= 0.0630vs^2/m
\end{aligned}
\tag{2}
$$

These parameters are obtained through reparameterization. There are no direct physical meanings. The units in the above parameters follow the International System for Units. The units arise because a fixed amplifier gain of K=1.2 A/v and a motor torque constant of 0.4006 Nm/A are took into account. Please refer to the manufacturer's user manual [1] for detail. For the convenience of future discussion, we call this Model One.

2.2 Equilibrium Configuration

An equilibrium configuration is a particular value of the state and the control input for which the Pendubot is at rest, i.e, $\dot{q} = 0$. Examining the equations in (1) of the Pendubot, the equilibrium points are given by

$$
\begin{aligned}
\theta_4 gcos(q_1) + \theta_5 gcos(q_1+q_2) &= \tau_{10} \\
\theta_5 gcos(q_1+q_2) &= 0
\end{aligned}
$$

Suppose $\frac{\tau_{10}}{\theta_4 g} \leq 1$, then solving for the equilibrium configuration

$$
q_1 = \arccos(\frac{\tau_{10}}{\theta_4 g}), q_2 = n\frac{\pi}{2} - q_1; \text{ n=1,3,5, ...}
$$

which means the Pendubot will balance at a state (q_1, q_2, 0, 0), if we apply a constant torque τ_{10}. The last two elements of the state are velocities.

We are interested in the natural equilibria of the Pendubot when $\tau_{10} = 0$. Examining the above solutions, we have the following four equilibrium configurations.

- $q_1 = -\frac{\pi}{2}$, $q_2 = 0$, (both link 1 and link 2 are in their lower positions).
- $q_1 = -\frac{\pi}{2}$, $q_2 = \pi$, (link 1 is in its lower position and link 2 is in its upper position).
- $q_1 = \frac{\pi}{2}$, $q_2 = 0$, (both link 1 and link 2 are in their upper positions).
- $q_1 = \frac{\pi}{2}$, $q_2 = \pi$, (link 1 is in its upper position and link 2 is in its lower position).

Note that only the first equilibrium point of the above four equilibria is stable. The remaining three equilibrium points are unstable. An arbitrary small disturbance causes at least one of the links to fall, and consequently a large motion is produced. Furthermore, the third equilibrium configuration $q_1 = \frac{\pi}{2}$, $q_2 = 0$ is the most difficult case for feedback stabilization, since very small disturbances will cause both links to fall.

2.3 Control Properties of the Pendubot

From equation (1), we see the dynamics of the Pendubot are subject to a second order differential constraint as follows

$$
\begin{aligned}
(\theta_2 + \theta_3 cos(q_2))\ddot{q}_1 &+ \theta_2\ddot{q}_2 + \theta_3 sin(q_2)\dot{q}_1^2 \\
&+ \theta_5 gcos(q_1+q_2) = 0
\end{aligned}
\tag{3}
$$

The integrability of dynamic constraints is an important property for many physical systems. Depending on the integrability of their dynamic constraints, dynamic systems can be classified as either holonomic or nonholonomic. A large class of under-actuated mechanical systems are second order nonholonomic systems. Controllability and stabilizability of under-actuated mechanical systems are closely related to integrability property. It is well known that a significant drift term exists for nonholonomic systems, it is difficult if not impossible to use smooth feedback to asymptotically stabilize a large class of nonholonomic systems to the equilibrium state. In this case, non-smooth feedback stabilization must be pursued or different control objectives must be addressed [4].

In order to check whether a system is holonomic or nonholonomic, integrability of the differential constraint needs to be checked. However, many integrability conditions in the literature can not be used for this purpose. They are either coordinate dependent or have strong assumptions.

New integrability conditions using the Frobenius Theorem in differential forms are presented in [5]. The conditions are coordinate independent and in general can be applied for any order of differential constraint. The conditions state a second order differential constraint is integrable if and only if the wedge product of ω and $d\omega$, where ω is the constraint in differential form, is vanishing.

Consider the second order differential constraint (3). After simple transformation, we obtain the following differential forms

$$\begin{aligned} \omega &= (\theta_2 + \theta_3 cos(q_2))d\dot{q}_1 + \theta_2 d\dot{q}_2 \\ &\quad + [\theta_3 sin(q_2)\dot{q}_1^2 + \theta_5 g cos(q_1 + q_2)]dt \\ d\omega &= d(\theta_2 + \theta_3 cos(q_2)) \wedge d\dot{q}_1 \\ &\quad + d[\theta_3 sin(q_2)\dot{q}_1^2 + \theta_5 g cos(q_1 + q_2)] \wedge dt \end{aligned}$$

It can be checked that $\omega \wedge d\omega \neq 0$.

We conclude that the Pendubot is a second order nonholonomic system.

Oriolo and Nakamura [7] showed that the dynamic constraint of an under-actuated two-link robot is holonomic if the gravity term vanishes and only the second link is controlled. If the first link is actuated, it is a second order nonholonomic system. For the Pendubot, not only is the first link actuated, but also the gravity term is non zero. Thus, it is a second order nonholonomic system. This observation is consistent with our conclusion using new integrability conditions in differential forms.

3 Hybrid Control for the Pendubot

In order to feedback stabilize the Pendubot around the equilibrium $q_1 = \frac{\pi}{2}$ and $q_2 = 0$ (both links are in their upper position), we need to move the Pendubot from its stable downward position (both links in their lower positions) to an unstable equilibrium manifold close to the inverted position . The strategy is to use swing up control first to move the Pendubot close to the equilibrium, then switch to a hybrid control for feedback stabilization.

3.1 Swing Up Control

Moving the Pendubot from its downward position to a neighborhood of its equilibrium manifold is called swing up control. Swing up has been fully studied in the literature. A good choice for swing up for the Pendubot is partial feedback linearization [1, 3].

It has been shown that the Pendubot dynamics are not feedback linearizable with static state feedback and nonlinear coordinate transformation [3]. However, we may achieve a linear response from either link, but not both, by suitable nonlinear partial feedback linearization.

Choose the control

$$\begin{aligned} \tau_1 &= \left(d_{11} - \frac{d_{12}d_{21}}{d_{22}}\right)u + \left(c_{11} - \frac{d_{12}c_{21}}{d_{22}}\right)\dot{q}_1 \\ &\quad + \left(c_{12} - \frac{d_{12}c_{22}}{d_{22}}\right)\dot{q}_2 + e_1(q) - \frac{d_{12}e_2(q)}{d_{22}} \end{aligned}$$

Then, the system in (1) is partially feedback linearized and u is an additional (outer loop) control input to be designed.

Choose the control u as

$$u = k_p(q_{e1} - q_1) - k_d\dot{q}_1 \quad (4)$$

It can be easily shown that if we choose $k_p > 0$ and $k_d > 0$ and suppose that the output q_1 identically tracks the equilibrium $q_{e1} = \frac{\pi}{2}$, then the linearized subsystem defines a globally attractive invariant manifold. The remaining nonlinear subsystem can be defined as the zero dynamics of the system with respect to the output q_1. The strategy for swing up control is to excite the zero dynamics sufficiently by the motion of link 1 so that the pendulum swings close to its unstable equilibrium manifold.

- For Model One supplied by the manufacturer [1], choose $k_p = 50/s^2$ and $k_d = 8.8/s$.
- For the purpose of robustness comparison, we also applied our hybrid control to the following model with varied parameters(called Model Two).

$$\begin{aligned} \theta_1 &= 0.0260vs^2, \quad \theta_2 = 0.0119vs^2 \\ \theta_3 &= 0.0098vs^2, \quad \theta_4 = 0.1673vs^2/m \\ \theta_5 &= 0.0643vs^2/m \end{aligned} \quad (5)$$

  For Model Two, we choose $k_p = 53/s^2$ and $k_d = 8.62/s$ for swing up control.

3.2 Feedback Stabilization

Once the Pendubot reaches the unstable equilibrium manifold, the controller will switch to a hybrid controller for asymptotically stabilizing the system to the equilibrium state.

A new hybrid control design theorem for feedback stabilization of a large class of under-actuated mechanical systems has been given in [5]. The hybrid control contains a continuous-time control part and a discrete-time control part. The continuous-time control part depends on continuous-time state information. The discrete-time control part changes values at every fixed time. The values are determined by both the previous discrete-time control part and the state information. Whenever the discrete-time control part switches to a new value, the discrete-time control part will always keep part of the previous control information. The hybrid controller design easily calculates matrices to satisfy the proper conditions.

Define the equilibrium point $q_e = [q_{e1}, q_{e2}]^t$, where q_{e1} is the actuated part and q_{e2} is the unactuated part. The system (1) is uniformly asymptotically stable to q_e,under the following hybrid control v, if there exists constants k_1, k_2, k_3, k_4, C, D and a positive real number $T \leq 1$, such that A is nonsingular and H has all its eigenvalues within the unit circle. Here, T represents the switching time for the discrete control $u(k)$.

$$
\begin{aligned}
v &= \bar{M}(q)[k_1(q_1 - q_{e1}) + k_2(q_2 - q_{e2}) \\
&\quad + k_3\dot{q}_1 + k_4\dot{q}_2 + u(k)] \\
&\quad + \bar{C}(q,\dot{q}) + \bar{e}(q) \\
u(k+1) &= Cu(k) + D \begin{bmatrix} q_1 - q_{e1} \\ q_2 - q_{e2} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\
\bar{M}(q) &= M_{11}(q) - M_{12}(q)M_{22}^{-1}(q)M_{21}(q) \\
\bar{C}(q,\dot{q}) &= C_{11}(q,\dot{q})\dot{q}_1 + C_{12}(q,\dot{q})\dot{q}_2 \\
&\quad - M_{12}(q)M_{22}^{-1}(q)[C_{21}(q,\dot{q})\dot{q}_1 \\
&\quad - C_{22}(q,\dot{q})\dot{q}_2] \\
\bar{e}(q) &= e_1(q) - M_{12}(q)M_{22}^{-1}(q)e_2(q) \\
H &= \begin{bmatrix} e^{AT} & A^{-1}(e^{AT} - I)B \\ D & C \end{bmatrix} \\
A &= \begin{bmatrix} 0 & 0 & I_{m\times m} & 0 \\ 0 & 0 & 0 & I_{(n-m)\times(n-m)} \\ k_1 & k_2 & k_3 & k_4 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ 0 \\ I_{m\times m} \\ -M_{22}^{-1}(q_e)M_{21}(q_e) \end{bmatrix} \\
A_{41} &= -M_{22}^{-1}(q_e)\left[M_{21}(q_e)k_1 + \frac{\partial e_2(q_1,q_2)}{\partial q_1}|_{q=q_e}\right] \\
A_{42} &= -M_{22}^{-1}(q_e)\left[M_{21}(q_e)k_2 + \frac{\partial e_2(q_1,q_2)}{\partial q_2}|_{q=q_e}\right] \\
A_{43} &= -M_{22}^{-1}(q_e)\left[M_{21}(q_e)k_3 + c_{21}(q_e)\right] \\
A_{44} &= -M_{22}^{-1}(q_e)\left[M_{21}(q_e)k_4 + c_{22}(q_e)\right]
\end{aligned}
$$

All the parameters are defined consistent with equation (1). For example, $M_{11}, M_{12}, M_{21} M_{22}$ are corresponding to the inertia matrix. One may use MAPLE or MATLAB to conduct the computation, and design the hybrid control parameters for asymptotically stabilizing the system to the equilibrium state.

- For Model One (refer to (2)), we choose

$$
\begin{aligned}
k_1 &= 2884.2, \quad k_2 = 2192.0, \quad k_3 = 479.5 \\
k_4 &= 282.9, \quad d_1 = 0.2, \quad d_2 = 0.3 \\
d_3 &= 0.1, \quad d_4 = 0.2, \quad c = 0.4, \quad T = 0.1
\end{aligned}
$$

- For Model Two (refer to (5)), we choose

$$
\begin{aligned}
k_1 &= 3224.5, \quad k_2 = 2463.0, \quad k_3 = 547.9 \\
k_4 &= 331.7, \quad d_1 = 0.2, \quad d_2 = 0.3 \\
d_3 &= 0.1, \quad d_4 = 0.2 \quad c = 0.4, \quad T = 0.1
\end{aligned}
$$

The reader may verify these parameters satisfy all the above conditions.

4 Implementation Results for the New Hybrid Control

We implemented the hybrid control algorithm for both Model One and Model Two. Some external disturbances were also added randomly by lightly hitting the links using a metal stick to test the robustness of the algorithm. Trajectories for both link one and link two are given for comparison.

Control Based on Model One: The following cases have been performed using hybrid control for Model One(as supplied by the manufacturer).

- Without disturbances: Figure 3 a and b show the positions and position errors of the two links.

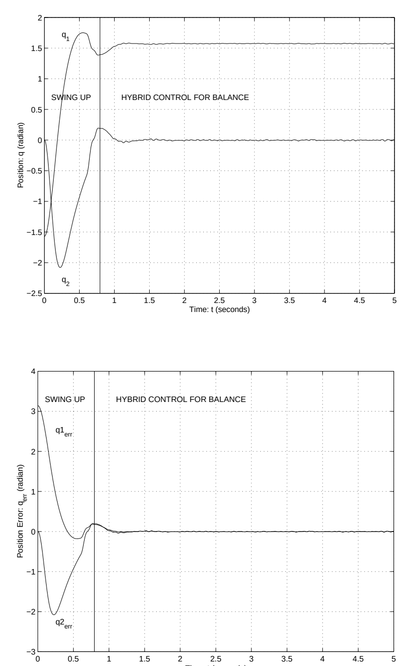

Figure 3: Without Disturbances

- With randomly added quick disturbances: Figure 4 shows the positions of the two links.

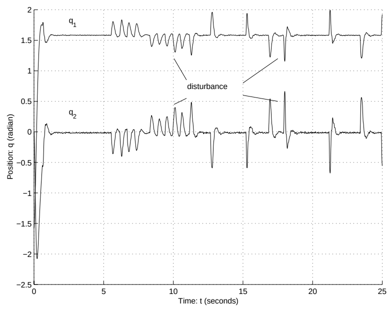

Figure 4: Randomly Added Quick Disturbances

- With randomly added slow disturbances for visually estimating the region of attraction: Figure 5 shows the positions of the two links.

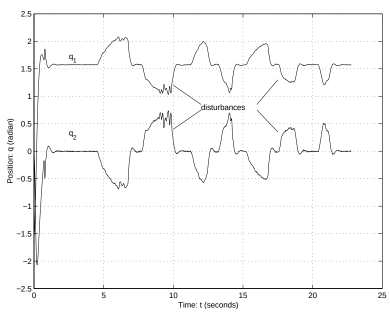

Figure 5: Randomly Added Slow Disturbances

Control Based on Model Two: Figures 6 a and b show the positions and position errors of the two links implementing hybrid control for Model Two.

5 Implementation Results for the Controller Supplied by the Manufacturer

The manufacturer [1] supplied controllers that use partial feedback linearization and nonlinear zero dynamics for swing up, and Linear Quadratic Optimal Control Theory to balance the Pendubot at the open loop unstable equilibrium $q_1 = \frac{\pi}{2}$ and $q_2 = 0$. Since the effect of friction in the motor brushes and bearings at the first joint and in the bearings at the second joint generally result in limit cycle behavior, the controller supplied by the manufacturer also includes a small dither signal that reduces the amplitude of the limit cycle. This reduction is called the friction compensation technique.

The Figures 7 a and b show positions and position errors of link one and link two.

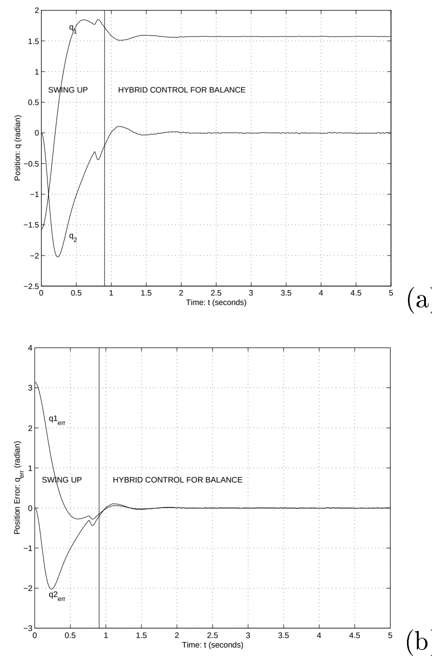

Figure 6: Varied Parameters Model

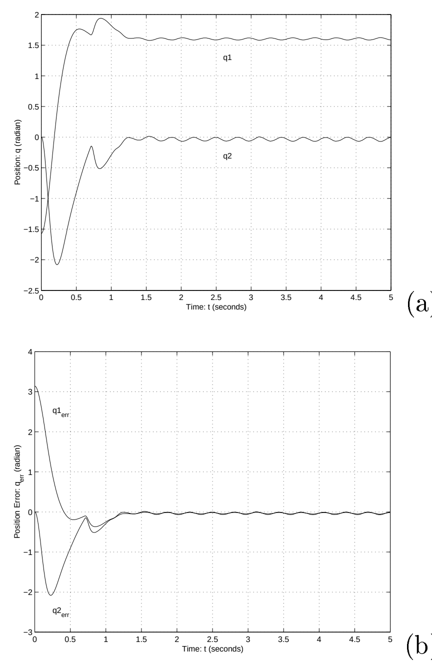

Figure 7: Manufacturer's Controller

6 Analysis and Comparison of the Experimental Results

From studying the above experimental results, we have come to the following conclusions.

- Implementation results show that the hybrid controller developed from [5] works very well. It shows a small transient shortly after the system is switched to hybrid control.

- Figure 4 shows that the hybrid controller responds very quickly and robustly, even to large uncertain disturbances.

- Figure 5 shows that the region of attraction is quite large for hybrid control. Slow uncertain disturbances were applied for estimating the region of attraction, which was found to be 72 degrees centered around the equilibrium state.

- The hybrid control is very robust with respect to the model variation. Figure 6 shows a somewhat surprising result: the hybrid control still works very well for large variations of the dynamic model. However, the manufacturer controller does not perform very well under these conditions. Our swing up controller can be switched to the hybrid controller much faster than the controller supplied by the manufacturer.

- Comparison of Figure 3 with Figure 7 shows that the hybrid controller works better than the controller supplied by the manufacturer. For the hybrid controller, both position and velocity errors are significantly smaller than that of the controller supplied by the manufacturer. Also, the response time is much faster than that of the controller supplied by the manufacturer. Once our controller switches to hybrid control, it quickly reaches steady state.

- The Pendubot remains a very special case among under-actuated mechanical systems that can use Linear Quadratic Optimal Theory for achieving smooth control. For a large class of under-actuated mechanical systems, it is impossible to use smooth feedback to asymptotically stabilize the system around the equilibrium state. For this reason, we developed the hybrid control technique, which can be used for designing a hybrid controller for a large class of under-actuated mechanical systems, especially where smooth feedback cannot be used to asymptotically stabilize the equilibrium state.

- We should note that we did not consider any technique for friction compensation in our hybrid control. Even though friction is very large and asymmetric for the Pendubot, the hybrid controller still outperforms the controller supplied by the manufacturer.

7 Conclusions

Experimental studies for control of nonholonomic systems have significant meanings for control of underactuated mechanical systems. The Pendubot, an underactuated mechanical system that shows second order nonholonomic properties, is an ideal test bed for this purpose.

This paper shows the new integrability conditions and hybrid control theorems for control of the Pendubot are a valuable approach for both theoretical and experimental study of control of second order nonholonomic systems.

Gravity terms make the Pendubot a special case for second order nonholonomic systems. If there were no gravity terms, it could not be feedback stabilized using smooth control, and the controller supplied by the manufacturer could not be used. The case for the omission of gravity terms can be easily found for airplanes, space craft, underwater manipulators,and underwater robotic vehicles and vessels. In such conditions, our hybrid control can be used. However, a practical implementation remains to be demonstrated.

8 Acknowledgement

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