Underactuated Mechanical Systems *

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Abstract

In this paper we discuss the control of underactuated mechanical systems. Underactuated mechanical systems have fewer control inputs than degrees of freedom and arise in applications, such as space and undersea robots, mobile robots, flexible robots, walking, brachiating, and gymnastic robots. The Lagrangian dynamics of these systems may contain feedforward nonlinearities, non-minimum phase zero dynamics, nonholonomic constraints, and other properties that place this class of systems at the forefront of research in nonlinear control [21, 13]. A complete understanding of the control of these systems is therefore lacking. We will discuss the application of geometric nonlinear control, as well as methods based on passivity and energy for stabilization and tracking control. We will survey some of the existing results and point to open research problems.

1 Introduction

A mechanical system may be "underactuated" in several ways. The most obvious way is from intentional design as in the brachiation robot of Fukuda [31], the passive walker of McGeer [22], the Acrobot [5], or the Pendubot [38]. Underactuated systems also arise in mobile robot systems, for example, when a manipulator arm is attached to a mobile platform, a space platform, or an undersea vehicle [45]. A third way that underactuated systems arise is due to the mathematical model used for control design as, for example, when joint flexibility is included in the model [39]. It is also interesting to note that certain control problems for fully actuated redundant robots are similar to those for underactuated robots [20]. The class of underactuated mechanical systems is thus rich in both applications and control problems.

The class of underactuated mechanical systems is far too broad to survey in a single article. For fully actuated systems there are a number of control results that apply to the entire class, such as feedback linearization and passivity-based adaptive control [40]. By contrast, with the exception of the collocated partial feedback linearization result discussed below, there are few results that are applicable to the entire class of underactuated mechanical systems. For example, the control

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problems for flexible joint robots require somewhat different tools for analysis and controller design than the control problems for gymnastic robots like the Acrobot.

In this article we will confine our discussion primarily to control problems for serial link robots containing both active and passive joints, such as the Acrobot [26, 4, 36] and Pendubot [38, 1]. Our ultimate goal for studying such systems is to understand problems of balance and locomotion in both biological systems and in robotic systems. The reader is referred to the literature for treatment of other classes of underactuated systems, such as flexible link robots [3], flexible joint robots [39], space robots [9], mobile robots [27] or underwater robots [10].

The techniques we will discuss for control are mainly based on ideas of passivity and control of energy. Passivity based control has a long and rich heritage having its roots in passive network synthesis and entering the control field via the Popov Criterion and the Kalman-Yakubovich-Popov Lemma [16]. Passivity in Lagrangian systems is equivalent to the now familiar skew-symmetry property [29], long known in classical mechanics and whose rediscovery in robot control led to breakthroughs in adaptive control of fully actuated manipulators [34]. In the nonlinear control field the exploitation of passivity has led to dramatic advances in controller design, with the appearance of concepts such as backstepping [17] and more recently forwarding [21, 33]. These methods are not yet generally applicable to all underactuated mechanical systems, but can be applied in special cases. In the area of robot locomotion, energy and passivity methods have already achieved some success. Indeed, the work of McGeer and others on passive walking [22] shows that stable limit cycle walking can be achieved by the natural tradeoff between kinetic and potential energy without feedback control of any kind. This work is fundamental, since, for example, there is considerable experimental evidence that a great part of the swing phase in human locomotion is passive. The muscles of the human leg are active primarily during the double support period, when the initial conditions on the angles and velocities of each of the limb segments are being established, after which they essentially turn off and allow the leg to swing through like a jointed pendulum [23]. This use of inertia and gravity coupled with the elastic energy stored and recovered from tendons, muscles, and bones, helps to account for the efficiency of animal locomotion.

2 Lagrangian Dynamics

For fully actuated mechanical systems a broad range of powerful techniques were developed in the last decade for the design of optimal, robust, adaptive, and learning controllers [40]. These techniques are possible because fully actuated systems possess a number of strong properties that facilitate control design, such as feedback linearizability, passivity, matching conditions, and linear parametrizability. For underactuated systems one or more of the above structural properties are usually lost. Moreover, undesirable properties such as higher relative degree and nonminimum phase behavior are manifested. For these reasons, control design becomes much more difficult and there are correspondingly fewer results available.

Consider the Lagrangian formulation of the dynamics of an n–degree-of-freedom mechanical system

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = B(q)\tau \qquad (1)$$

where $q \in R^n$ is the vector of generalized coordinates, $\tau \in R^m$ is the input generalized force $(m < n)$, and $B(q) \in R^{n \times m}$ has full rank for all q.

For a suitable partition of the vector q of generalized coordinates as $q^T = (q_1^T, q_2^T)$, where $q_1 \in R^{n-m}$ and $q_2 \in R^m$ we may write the system (1) as

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1(q_1, \dot{q}_1, q_2, \dot{q}_2) + \phi_1(q_1, q_2) &= 0 && (2) \\
d_{12}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2(q_1, \dot{q}_1, q_2, \dot{q}_2) + \phi_2(q_1, q_2) &= b(q_1, q_2)\tau && (3)
\end{aligned}
$$

where h_i include Coriolis and centrifugal terms, and ϕ_i contains the terms derived from the potential energy, such as gravitational and elastic generalized forces. The $m \times m$ matrix $b(q_1, q_2)$ is assumed to be invertible.

Example 1: Two Link Robot. Consider the two-link robot shown in Figure (1):

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \phi_1 &= \tau_1 && (4) \\
d_{12}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \phi_2 &= \tau_2 && (5)
\end{aligned}
$$

where

$$
\begin{aligned}
d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos(q_2)) + I_1 + I_2 \\
d_{22} &= m_2\ell_{c2}^2 + I_2 \\
d_{12} &= m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos(q_2)) + I_2 \\
h_1 &= -m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2^2 - 2m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2\dot{q}_1 \\
h_2 &= m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_1^2 \\
\phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos(q_1) + m_2\ell_{c2}g\cos(q_1 + q_2) \\
\phi_2 &= m_2\ell_{c2}g\cos(q_1 + q_2)
\end{aligned}
$$

If $\tau_1 = 0$ this system represents the Acrobot [26, 5], while if $\tau_2 = 0$ the system represents the Pendubot [38]. In addition, with $\phi_1 = 0 = \phi_2$ and $\tau_2 = 0$ one has the underactuated manipulator system considered in by several authors, such as [28, 7, 2].

Example 2: Cart-Pole System. The cart-pole system is one of the classic examples and yet it still holds some interesting challenges from the standpoint of global nonlinear control. Referring to Figure (2) the dynamics are given by:

$$
\begin{aligned}
(m_p + m_c)\ddot{x} + m_p\ell\cos\theta\ddot{\theta} - m_c\dot{\theta}^2\sin\theta &= F \\
m_p\ell\cos\theta\ddot{x} + m_p\ddot{\theta} - m_p\ell g\sin\theta &= 0
\end{aligned}
$$

For simplicity we normalize all constants to unity. To put the system in standard form, we set $q_1 = \theta$, $q_2 = x$, $\tau = F$, and write the equations as

$$
\begin{aligned}
\ddot{q}_1 + \cos q_1\ddot{q}_2 - \sin q_1 &= 0 && (6) \\
\cos q_1\ddot{q}_1 + 2\ddot{q}_2 - \dot{q}_1^2\sin q_1 &= \tau && (7)
\end{aligned}
$$

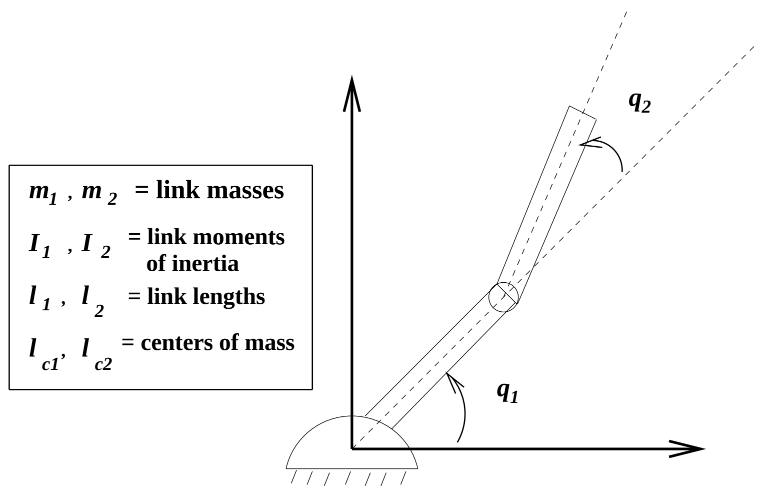

Figure 1: Two Link Robot

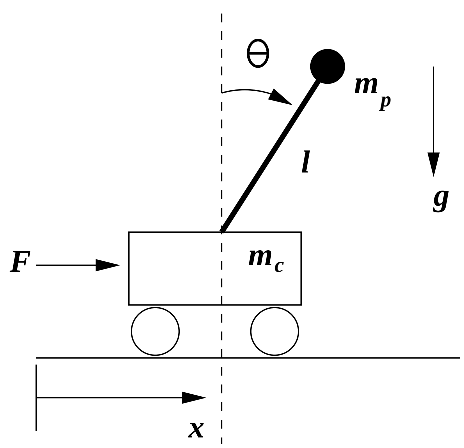

Figure 2: Cart-Pole System

2.1 Equilibrium Solutions and Controllability

The nature of the fixed points of (2)-(3) is closely tied to the controllability of the system. Let $\tau = \bar{\tau} =$ constant. Then, since the terms h_i are quadratic in the velocities $\dot{q}_i$, the equilibrium solutions satisfy

$$\begin{aligned} \phi_1(q_1, q_2) &= 0 && (8) \\ \phi_2(q_1, q_2) &= b(q_1, q_2)\bar{\tau} && (9) \end{aligned}$$

and may either be isolated fixed points for each fixed $\bar{\tau}$, as in the case of the Acrobot, and Pendubot or they may be higher dimension as happens (for $\bar{\tau} = 0$) in systems without potential terms. For example, in the absense of gravity, the Pendubot dynamics satisfies

$$\phi_i(q_1, q_2) = 0 \qquad \text{for } i = 1, 2 \tag{10}$$

for all $(q_1, q_2) \in Q$, where Q denotes the two dimensional configuration space.

In the first case, systems with potential terms are linearly controllable around (almost all) fixed points, i.e. the Taylor series linearization is a controllable linear system. Systems without potential terms are generally not linearly controllable. Their local controllability properties are therefore more subtle to determine.

We may interpret equation (2) as a (dynamic) constraint on the accelerations of the generalized coordinates. It is then interesting to ask whether these constraints are holonomic, i.e. integrable. For many of the most interesting cases, including underactuated robot manipulators [28], the Acrobot [36] and Pendubot [38], the PVTOL system [43], the TORA system, and underwater robots, these constraints turn out to be completely nonintegrable as shown in [30]. An important consequence is that the system (2)-(3) is (strongly) accessible, since nonintegrability of the second order constraint equations means that the dimension of the reachable set is not reduced.

Accessibility does not imply stabilizability of an equilibrium configuration using time-invariant continuous state feedback (either static or dynamic). In fact, for systems without potential terms it is known [30] that such stabilizability is not possible. The proof of this follows from an application of Brockett's Theorem [6]. The situation here is, therefore, quite similar to the case of control of nonholonomic mobile robots. Of course, systems with potential terms are exponentially stabilizable by linear time-invariant feedback.

3 Partial Feedback Linearization

An interesting property that holds for the entire class of underactuated mechanical systems is the so-called collocated partial feedback linearization property [35], which is a consequence of positive definiteness of the inertia matrix. A related property, the non-collocated partial feedback linearization, holds for a restricted class of underactuated systems.

3.1 Collocated Linearization

Collocated linearization refers to a control that linearizes the equations associated with the actuated degrees of freedom q_2. Equivalently, collocated linearization can be thought of as input/output linearization [12] with respect to an output $y = q_2$. The result states that the original system (2)-(3) is feedback equivalent to the system

$$
\begin{aligned}
d_{11}\ddot{q}_1 + h_1 + \phi_1 &= -d_{12}u && (11)\\
\ddot{q}_2 &= u && (12)
\end{aligned}
$$

with a suitable nonlinear feedback control

$$\tau = \alpha(q_1, \dot{q}_1, q_2, \dot{q}_2) + \beta(q_1, q_2)u \qquad (13)$$

where u is a new control input to be determined. The derivation is straightforward and is contained in [35].

3.2 NonCollocated Linearization

Noncollocated linearization refers to linearizing the passive degrees of freedom and is possible under a special assumption on the inertia matrix of the robot.

Definition 3.1 *The system (2)-(3) is (locally) Strongly Inertially Coupled if and only if*

$$rank(d_{12}(q)) = n - m \qquad \text{for all } q \in \mathcal{B}$$

where $\mathcal{B}$ is a neighborhood of the origin. The Strong Inertial Coupling is global if the rank condition holds for all $q \in Q$.

Note that Strong Inertial Coupling requires $m \geq n - m$, i.e. that the number of active degrees of freedom be at least as great as the number of passive degrees of freedom. Under the assumption of Strong Inertial Coupling we may compute a pseudo–inverse $d_{12}^{\dagger}$ for d_{12} as

$$d_{12}^{\dagger} = d_{12}^{T}(d_{12}d_{12}^{T})^{-1}$$

and show the existence of a feedback control τ that transforms the system into the following feedback equivalent

$$
\begin{aligned}
\ddot{q}_1 &= u\\
\ddot{q}_2 &= -d_{12}^{\dagger}(d_{11}u + h_1 + \phi_1)
\end{aligned}
$$

The details are contained in [35]. We note that a system satisfying the (local) Strong Inertial Coupling Property is known as an *Internal/External Convertible* system in the terminology of Getz [11].

Example: The Cart-Pole System. The cart-pole system,

$$\begin{aligned}\ddot{q}_1 + \cos q_1 \ddot{q}_2 - \sin q_1 &= 0 \\ \cos q_1 \ddot{q}_1 + 2\ddot{q}_2 - \dot{q}_1^2 \sin q_1 &= \tau\end{aligned}$$

is strongly inertially coupled for $-\pi/2 < q_1 < \pi/2$ but not strongly inertially coupled globally. For collocated linearization it is easy to show that the control law

$$\tau = (2 - \cos^2 q_2)u + \cos q_1 \sin q_1 - \dot{q}_1^2 \sin q_1 \tag{14}$$

results in the feedback equivalent system

$$\begin{aligned}\ddot{q}_1 &= \sin q_1 - \cos q_1 u \\ \ddot{q}_2 &= u\end{aligned}$$

which is valid globally, while the control law

$$\tau = 2\tan q_1 - \dot{q}_1^2 \sin q_1 - \frac{1 + \sin^2 q_1}{\cos q_1} u \tag{15}$$

results in the feedback equivalent system

$$\begin{aligned}\ddot{q}_1 &= u \\ \ddot{q}_2 &= \tan q_1 - \frac{1}{\cos q_1} u\end{aligned}$$

valid for $q_1 \in (-\pi/2, \pi/2)$.

4 Cascade Systems

The advantages of the first stage partial feedback linearization are both a conceptual and a structural simplification of the control problem. We can write the systems under consideration, after the first stage partial feedback linearization, as

$$\dot{x} = Ax + Bu \tag{16}$$

$$\dot{\eta} = w(\eta) + h(\eta, x) + g(\eta, x)u \tag{17}$$

with suitable definitions of all quantities, such that $h(\eta, 0) = 0$. The pair (A, B) is controllable since the linear system is a set of m double integrators and the expression

$$\dot{\eta} = w(\eta) \tag{18}$$

represents the zero dynamics [12]. If the control term u is chosen to be a function only of x, for example $u = -Kx$, then the system will be in cascade form

$$\dot{x} = \bar{A}x \tag{19}$$

$$\dot{\eta} = \bar{w}(\eta, x) \tag{20}$$

where $\bar{A} = A - BK$ is a Hurwitz matrix and $\bar{w}(\eta, x) = w(\eta) + h(\eta, x) - g(\eta, x)Kx$. There are a number of local and global stabilization results for special classes of such cascade systems. Both the nature of the equilibrium solution of the zero dynamics and the nature of the coupling between the x and η subsystems determines the type of results that can be proven. See [32] for a detailed treatment of the latest results.

4.1 Passivity and Energy Control

For general nonlinear systems of the form (19)-(20) local asymptotic stability of the origin is guaranteed if the origin of the zero dynamics is locally asymptotically stable. Such systems are called minimum phase. Global stability requires consideration of issues such as peaking [41]. The systems considered here generally have multiple equilibrium points and are non-minimum phase in a neighborhood of a typical equilibrium point. Nevertheless we can utilize the special structure of the system (as a Lagrangian system) to show global stability in some cases. The crucial fact is the following,

Theorem 4.1 *Given the Lagrangian mechanical system (2)-(3), the zero dynamics of the collocated feedback equivalent system (11)-(12), equivalently (18), also defines a Lagrangian system, in particular, there exists a positive definite scalar (energy) function, $E(\eta)$, such that*

$$L_w E = 0 \tag{21}$$

The proof of this theorem is straightforward and is omitted. Mainly one needs to show that the kinetic energy of the original system is positive definite when restricted to the zero dynamics manifold $x = 0$ of (19)-(20). It is interesting to note that, in the case of noncollocated linearization, the zero dynamics fails to be a Lagrangian system. However, in some cases a Lyapunov-like function E may still be found satisfying (21). The importance of this result is that it can be used to ensure stability of the interconnection.

Consider a slightly simplified (single input) system in the form

$$\begin{aligned} \dot{x} &= Ax + Bu && (22) \\ \dot{\eta} &= w(\eta) + g(\eta, x)u && (23) \end{aligned}$$

Since $L_w E = 0$ we have

$$\dot{E} = L_g E u \tag{24}$$

which implies that (23) defines a passive system with respect to the input u and output $y_\eta = L_g E$. If we therefore choose $u = -Kx$ in (22) so that the transfer function $K(sI - A)^{-1}B$ is Strictly Positive Real (SPR), i.e. so that (16) is passive with respect to the output $y_1 = Kx$, then (22)-(23) can be represented as a feedback interconnection of passive systems and is therefore passive. Stability of the interconnection follows from an additional (detectability) assumption [32]. The trajectory of the system will, in fact, converge to a particular energy level, which corresponds to a particular trajectory on the zero dynamics manifold. This idea has been used to design swingup controllers for systems like the cart-pole, Acrobot, and Pendubot [37].

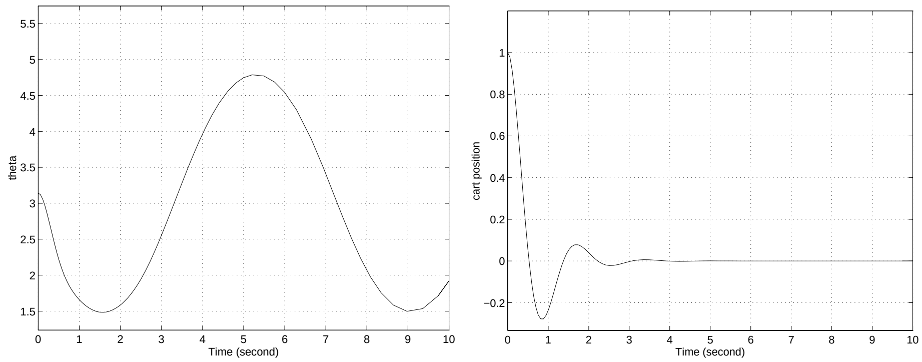

Figure 3: Response of the Cascade Cart-Pole System

Example: Swingup Control. Applying this control to the cart-pole system results in the response shown in Figure (3). Note that asymptotic stability is only guaranteed to a manifold and not to a fixed point. For this reason, control must eventually switch to a second control that locally stabilizes the equilibrium point. However, the control design is very simple and widely applicable as one way to overcome problems with feedforward nonlinearities and non-minimun phase zero dynamics [24, 36, 38].

4.2 Lyapunov Functions and Forwarding

Important extensions of the proceeding ideas are due to Teel [44], Mazenc and Praly [21], Sepulchre, Janković, and Kokotović [14, 33] and others who provide constructive procedures for global and semi-global asymptotic stabilization to a fixed point, rather than just to a manifold, for restricted subclasses of underactuated systems. A detailed discussion of these results is outside the scope of this article. The reader is referred to [32] and the references therein for details.

As a brief glimpse into one such approach we can illustrate the basic idea of the method of *Forwarding* [33]. Suppose that a Lyapunov function V_0 for the zero dynamics (18) is known and satisfies (21). Since $\bar{A} = A - BK$ is Hurwitz, $V_1 = x^T P x$ defines a Lyapunov function for (19) where P satisfies a Lyapunov equation for $\bar{A}$. The techniques in [32] provide procedures for constructing cross terms $\Phi(\eta, x)$ such that

$$V(\eta, x) = V_0(\eta) + \Phi(\eta, x) + V_1(x) \tag{25}$$

defines a Lyapunov function for the system (19)-(20). Calculating $\dot{V}$ along trajectories of (19)-(20) gives

$$\dot{V} = L_w V_0 + L_{h-gKx} V_0 + \dot{\Phi} + L_{\bar{A}x} V_1 \tag{26}$$

The trick is now to show that there exists Φ satisfying

$$\dot{\Phi} = -L_{h-gKx} V_0 \tag{27}$$

that simultaneously guarantees the required properties for V to be a (radially unbounded) Lyapunov function for the system. If such an expression can be found then global stability of (19)-(20) is assured. One may now go back and augment the control input u as

$$u = -Kx + v \tag{28}$$

and consider the system

$$\begin{aligned} \dot{x} &= \bar{A}x + Bv & (29) \\ \dot{\eta} &= \bar{w}(\eta, x) + g(\eta, x)v & (30) \end{aligned}$$

This system is of the form

$$\dot{z} = F(z) + G(z)v \tag{31}$$

where a Lyapunov function V is known satisfying $L_F V \leq 0$. It then follows that a Jurdjevic-Quinn type of control [15]

$$v = -L_G V \tag{32}$$

can be used to achieve global stability and, under further restrictions, global asymptotic stability.

4.3 Hybrid and Switching Control

The ideas in the previous section are applicable only to restricted classes of mechanical systems. The same is true for other techniques, such as backstepping [18], the technique of adding integrators due to Mazenc and Praly [21], and the saturation approach of Teel [43]. In the case of backstepping the system state equations must have a lower triangular structure while for forwarding the state equations must have an upper triangular structure. Even when applicable these methods lead to designs which can be computationally difficult. For example, the computation of the cross term Φ in (25) is possible only in simple examples.

One way to avoid these computational difficulties is to consider a hybrid control architecture that switches among several controllers, each of which may be simpler to design. For example, the global stabilization of the inverted position of the Pendubot using a single smooth controller designed using integrator forwarding is currently not possible. However, since the Pendubot is linearly controllable in a neighborhood of the inverted configuration, one need only design a nonlinear controller so that the trajectory intersects a suitable neighborhood of the desired equilibrium (swingup control) and then switch to a linear controller to stabilize the system around the equilibrium (balance control).

As an added benefit, the design of switching controllers in the context of locomotion is likely to lead to an improved understanding of locomotion in biological systems. The problem of locomotion while maintaining balance encompasses the transition from standing to walking and back to standing, as well as the transition among various gaits of locomotion. For example, a human is constantly starting, stopping, performing tasks while standing, sitting down, standing up, etc. Such a complex behavior cannot be achieved with a single smooth controller but may be achieved by switching among multiple controllers.

Consider a supervisory control architecture shown in Figure (4) for the problem of swingup and balance of a gymnastic robot. The Supervisor switches between a nonlinear Swing Up Controller

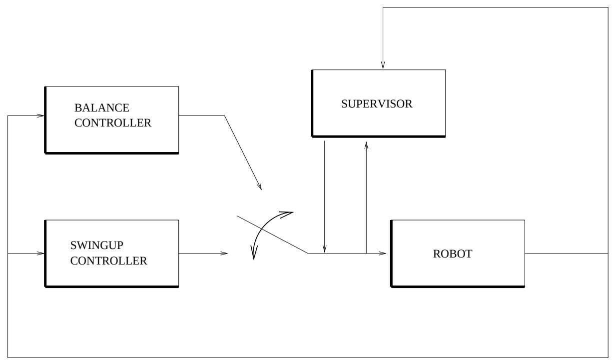

Figure 4: Supervisory Control Architecture

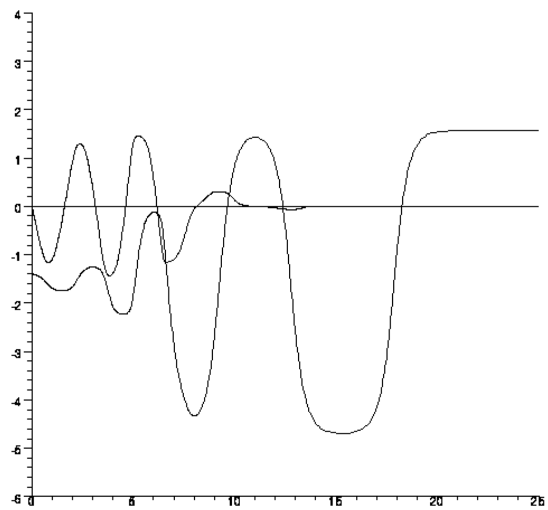

Figure 5: Swingup and Balance Control for the Acrobot

and a linear Balance Controller when the trajectory enters the basin of attraction of the local balancing controller. This architecture also allows for robustness to disturbances. Since the balance controller is only local, a large disturbance is handled by switching back into the swing up mode and re-converging to the basin of attraction of the balance controller. A successful swingup and balance control for the Acrobot is shown in Figure (5).

4.4 Nonholonomic Systems

Underactuated systems which are not linearly controllable, typically those without gravitational or elastic terms in the dynamics, are amenable to motion planning and control approaches similar to those that used for mobile robots. Consider the Pendubot in the absense of gravity, which after partial feedback linearization, can be expressed in the form [7]

$$\ddot{q}_1 = u \tag{33}$$

$$\ddot{q}_2 = -c \sin q_2 \dot{q}_1^2 - (1 + c \cos q_2) u \tag{34}$$

The main difference between systems, such as the above, with acceleration constraints and mobile robots with velocity constraints is the presence of the drift term in the equations of motion, which complicates the controllability analysis. For example, for systems without drift, accessibility (in the sense of full rank of the accessibility distribution [12]) implies controllability by Chow's Theorem [25]. This is no longer true for systems with drift.

A stronger notion of controllability for underactuated systems is the property of small time local controllability (STLC) [42]. A sufficient condition for STLC was given for underactuated mechanical systems in [8]. This result is important since it implies the existence of either discontinuous or time periodic feedback controllers to stabilize the system to a point. Algorithms for point to point control of the above system are given in [7, 30].

5 Conclusions

Underactuated mechanical systems of the type considered here present many challenging opportunities for future research.

1. **Robust and Adaptive Control:** It is well known that the Lagrangian dynamic equations of robots are linear in the inertia parameters and that this linearity is generally lost when the system is written in state space. For fully actuated systems, the passivity based adaptive control [29] circumvents this difficulty. However, for underactuated systems, passivity is lost. The recursive design techniques of integrator backstepping and integrator forwarding are the proper extensions of passivity based design techniques when they are applicable. At the present time these techniques are applicable for those systems that retain linearity in the parameters and which satisfy certain structural properties, growth conditions on the nonlinear coupling, etc. The extension of these methods for robust and adaptive control of larger classes of underactuated systems is thus a research problem of major importance.

2. **Saturation Methods:** The method of Teel [43] using saturation functions is a powerful technique for achieving semi-global and global stabilization results. As in the case of backstepping and forwarding, results exist only for restricted classes of systems. Extending these methods to larger classes of underactuated systems is an important problem.

3. **Stability of Hybrid Systems:** The research problems in the use of hybrid and logic-based switching control for underactuated systems are mainly at the supervisory level, i.e. determining when to switch and proving stability. This turns out to be highly non-trivial. Formal stability results exist only for limited classes of hybrid systems. It is, in fact, known that stability results for the general class of hybrid systems cannot be obtained. This is because the vocabulary for describing hybrid systems is too expressive to permit such strong results. One can embed a universal Turing machine into a hybrid system so that the stability question can reduce, in the worst case, to the *halting problem* [19], which is known to be undecidable. Thus one necessarily must focus on specific classes of hybrid systems in order to make progress.

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