EXAMPLE FINAL EXAM

INSTRUCTIONS

This exam is open book and open notes. It consists of 4 problems and is worth a total of 160 points. The problems are not of equal difficulty, so use discretion in allocating your time. Attempt to answer all questions in any order.


1. SUFFICIENT STATISTICS AND ML ESTIMATION (40 points)

Suppose that \{X_1, X_2, \ldots, X_N\} is a random sample drawn according to the following probability mass function (pmf):

\[
p_X(x) = \binom{2}{x} \theta^x (1 - \theta)^{2-x} I_{\{0,1,2\}}(x),
\]

where \(I(x)\) is the indicator function and \(\theta > 0\) is an unknown (nonrandom) parameter.

(a) Find a one-dimensional sufficient statistic.

(b) This statistic is obviously minimal sufficient. Determine if it is also complete. (Hint: Note that the pmf of the sufficient statistic is similar to that in (1), except that it has parameters \(\{\theta, 2N\}\).)

(c) Find the maximum-likelihood estimators for \(P[X_i = 0] = \theta\) and \(P[X_i = 2] = \theta^2\).

(d) Determine if these estimators are unbiased.
2. KALMAN PREDICTOR (40 points)

When the measurement noise \( v(k) \) and the unmeasurable disturbance \( w(k) \) are correlated (with cross-correlation matrix \( S \)), the one-step Kalman predictor is given by

\[
\hat{x}(k+1|k) = \Phi \hat{x}(k|k-1) + \Psi u(k) + L(k) \tilde{z}(k|k-1)
\]

\[
L(k) = [\Phi P(k|k-1)H^T + \Gamma S] \times [HP(k|k-1)H^T + R]^{-1}
\]

\[
P(k+1|k) = [\Phi - L(k)H]P(k|k-1)[\Phi - L(k)H]^T + \Gamma QF^T - \Gamma SL^T(k) - L(k)ST^T + L(k)RL^T(k)
\]

where \( L(k) \) is the Kalman predictor gain, \( \tilde{z}(k|k-1) \) is the usual prediction error, and we have assumed that the state-variable parameters are time invariant. Note that if \( S = 0 \), these equations reduce to the standard Kalman predictor covered in class.

Consider the following scalar measurable process:

\[
z(k) = v(k) + w(k-1)
\]

where \( v(k) = w(k) \) is a stationary white-noise sequence with zero mean and unit variance.

(a) For this process, determine the parameters of the first-order (scalar) state-variable model, i.e., specify \( \Phi, \Psi, \Gamma, H \), and the variances \( Q \) and \( R \).

(b) For the system in (5) and from your answer in part (a), show that the one-step Kalman predictor of \( z(k+1) \) is

\[
\hat{z}(k+1|k) = L(k)[z(k) - \hat{z}(k|k-1)].
\]

(c) For \( S = 1 \), show that (3) and (4) reduce to

\[
L(k) = [P(k|k-1) + 1]^{-1} \quad \text{and} \quad P(k+1|k) = 1 - L(k).
\]

(d) Let \( B(k) = P^{-1}(k|k-1) \) and show that, with the initial condition \( B(0) = 1 \), \( L(k) \) in (6) is

\[
L(k) = \frac{k + 1}{k + 2}.
\]
3. UNBIASED ESTIMATION AND MAP ESTIMATION (40 points)

Suppose that \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are two independent unbiased estimators of \( \theta \) with variances \( \sigma^2_1 \) and \( \sigma^2_2 \), respectively. Consider the new estimator

\[
\hat{\theta} = \alpha \hat{\theta}_1 + \beta \hat{\theta}_2
\]

where \( \alpha \) and \( \beta \) are nonrandom constants.

(a) Find \( \alpha \) and \( \beta \) such that the variance of \( \hat{\theta} \) is minimum and \( \hat{\theta} \) is unbiased.

(b) What is the corresponding minimum value of the variance?

(c) What are the optimal values of \( \alpha \) and \( \beta \) as \( \sigma^2_2 \to \infty \)? What is the corresponding minimum value of the variance?

Consider now the random variable \( Z \) which has a Poisson distribution with mean \( \theta \):

\[
f(z) = \frac{\theta^z}{z!} e^{-\theta} I_{\{0,1,\ldots\}}(z)
\]

where \( \theta \) is also a random variable with the following exponential distribution:

\[
f(\theta) = \lambda e^{-\lambda \theta} I_{[0,\infty)}(\theta)
\]

where the constant parameter \( \lambda > 0 \) is known. (Note that parts (d) and (e) below are independent of parts (a)–(c) above.)

(d) Find the maximum a posteriori (MAP) estimator of \( \theta \) from the single measurement \( Z \).

(e) How does your answer compare to the maximum likelihood (ML) estimator of \( \theta \)?
4. MS ESTIMATION (40 points)

Suppose that $X$ and $Y$ are correlated random variables with the following density functions:

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-y-y^2}{2} \right)^2}$$

$$f_Y(y) = e^{-y}I_{[0,\infty)}.$$

Note that since $Y$ is exponential with parameter $\lambda = 1$, the moments are $E(Y^k) = k!$.

(a) Find the nonlinear minimum mean-square (MS) estimator $\hat{X}$ of $X$ given that we observe $Y$.

(b) From your answer in part (a), determine the corresponding minimum value of the MS error.

(c) Find the linear minimum MS estimator of $X$ given that we observe $Y$. Your answer should be of the form $\hat{X} = aY + b$ where $a$ and $b$ are constants to be determined.

(d) From your answer in part (c), determine the corresponding minimum value of the MS error.

(Hint: Suppose we know $E(U|V)$. Then $E(U) = E_V[E(U|V)]$ where the outer expectation is over $V$.)