ECE 242
Gibson

Spring 2011
05/10/11

Midterm Exam Solutions

1. (a) From class, $D \geq \frac{1}{12} e^{-2(H(Y)-h(X))}$ where $h(X) = \log_2 e\sqrt{2} = 1.943$ and $H(Y) = 3$, so $D \geq 0.01$ and $SNR \leq 20\,\text{dB}$.

(b) $D(3) = \frac{e}{\pi} 2^{-6} = 0.0135$ or $SNR = 18.69\,\text{dB}$.

2. (a) The output $SNR = 10\log_{10}\frac{\sigma^2}{D}$ where $D = \frac{x_{\max}^2}{3N^2}\int_{-x_{\max}}^{x_{\max}} f_X(x)\left(\frac{d}{dx}G(x)\right)^{-2} dx$ and the pdf is Laplacian so $f_X(x) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x|/\sigma}$. Substitute $\frac{d}{dx}G(x) = \frac{m}{1-e^{-m}} e^{-m|x|/x_{\max}}$ and evaluate.

(b) $\Delta = \frac{2x_{\max}}{N}\left(\frac{d}{dx}G(x)\right)^{-1}$ so find $\Delta_{\min}$ by letting $x \to 0$ and $\Delta_{\max}$ by letting $x \to x_{\max}$ so we get $\frac{\Delta_{\max}}{\Delta_{\min}} = e^m$

3. (a) For the first quantizer, we input X, $\hat{X} = Q_1(X)$ and $E(X-\hat{X})^2 = EX^2 - E\hat{X}^2$. For the second quantizer, the input is $U = X - \hat{X}$, so $\hat{U} = Q_2(U)$ and we have $E(U-\hat{U})^2 = EU^2 - E\hat{U}^2 = EX^2 - E\hat{X}^2 - E\hat{U}^2$. Overall,
$E(X-\tilde{X})^2 = E(X-\hat{X}-\hat{U})^2 = E(U-\hat{U})^2$

(b) For optimal MMSE Q_2, $E(U-\hat{U})\hat{U} = 0 \Rightarrow EU\hat{U} = E\hat{U}^2$

4. See next page and following.

5. See next page and following.

4.] $D = E\{|X-Y|^r\} = \int_{-\infty}^{\infty} |x-y|^r p(x)\,dx$

$$= \sum_{i=1}^{L} \int_{x_i}^{x_{i+1}} |x-y_i|^r \frac{1}{V}\,dx =$$

$$\sum_{i=1}^{L} \left\{ \int_{x_i}^{y_i} (y_i-x)^r \frac{1}{V}\,dx + \int_{y_i}^{x_{i+1}} (x-y_i)^r \frac{1}{V}\,dx \right\}$$

(a) Necessary conditions: $\frac{\partial D}{\partial x_j} = 0$, $j = 2, \dots, L$

so

$$-(y_j - x_j)^r \frac{1}{V} + (x_j - y_{j-1})^r \frac{1}{V} = 0$$

or $y_j - x_j = x_j - y_{j-1} \Rightarrow x_j = \frac{y_j + y_{j-1}}{2}$.

$$\frac{\partial D}{\partial y_j} = 0 \Rightarrow (y_j - x)^r \frac{1}{V}\Big|_{x=y_j} + \int_{x_j}^{y_j} r(y_j-x)^{r-1} \frac{1}{V}\,dx$$

$$+(-1)(x-y_j)^r\Big|_{x=y_j} \frac{1}{V} + \int_{y_j}^{x_{j+1}} (-1)(x-y_j)^{r-1} \frac{1}{V}\,dx = 0$$

$$\therefore \int_{x_j}^{y_j} (y_j - x)^{r-1}\,dx = \int_{y_j}^{x_{j+1}} (x-y_j)^{r-1}\,dx$$

or $-\frac{(y_j-x)^r}{r}\Big|_{x_j}^{y_j} = \frac{(x-y_j)^r}{r}\Big|_{y_j}^{x_{j+1}}$

Thus, $(y_j-x_j)^r = (x_{j+1}-y_j)^r$

$\therefore\ y_j - x_j = x_{j+1} - y_j \Rightarrow y_j = \frac{x_{j+1}+x_j}{2}$.

$x_j = \frac{1}{2}\left\{\frac{x_{j+1}+x_j}{2} + \frac{x_j+x_{j-1}}{2}\right\} = \frac{1}{2}\left\{\frac{x_{j+1}}{2} + x_j + \frac{x_{j-1}}{2}\right\}$

or $x_j = \frac{x_{j+1}+x_{j-1}}{2}$

Similarly, $y_j = \frac{y_{j+1}+y_{j-1}}{2}$, so the quantizer is UNIFORM for all $r \geq 1$.

$x_1 = \frac{-V}{2}$, $x_{L+1} = \frac{V}{2}$, $x_j = -\frac{V}{2} + (j-1)\Delta$,

$y_j = \frac{-V}{2} + \frac{\Delta}{2} + (j-1)\Delta$, $\Delta = \frac{V}{L}$.

(b) $D = \sum_{i=1}^{L} \int_{x_i}^{x_{i+1}} |x-y_i|^r \frac{1}{V}\, dx =$

$$\sum_{i=1}^{L}\left\{\int_{x_i}^{y_i}(y_i-x)^r\frac{1}{V}dx+\int_{y_i}^{x_{i+1}}(x-y_i)^r\frac{1}{V}dx\right\}$$

$$=L\left\{\int_0^{\Delta/2}\left(\frac{\Delta}{2}-x\right)^r\frac{1}{V}dx+\int_{\Delta/2}^{\Delta}\left(x-\frac{\Delta}{2}\right)^r\frac{1}{V}dx\right\}$$

$$=\frac{L}{V}\left\{\left.\frac{-\left(\frac{\Delta}{2}-x\right)^{r+1}}{r+1}\right|_0^{\Delta/2}+\left.\frac{\left(x-\frac{\Delta}{2}\right)^{r+1}}{r+1}\right|_{\Delta/2}^{\Delta}\right\}$$

$$=\frac{L}{V}\left\{\frac{\left(\frac{\Delta}{2}\right)^{r+1}}{r+1}+\frac{\left(\frac{\Delta}{2}\right)^{r+1}}{r+1}\right\}=\frac{2L}{V}\frac{\left(\frac{\Delta}{2}\right)^{r+1}}{r+1}$$

But $\Delta = V/L$ and $L = 2^R$, so $\Delta = V2^{-R}$

and

$$D=\frac{2\cdot 2^R}{V}\cdot\frac{V^{r+1}2^{-R(r+1)}}{2^{r+1}(r+1)}=\frac{V^r}{r+1}2^{-r(R+1)}$$

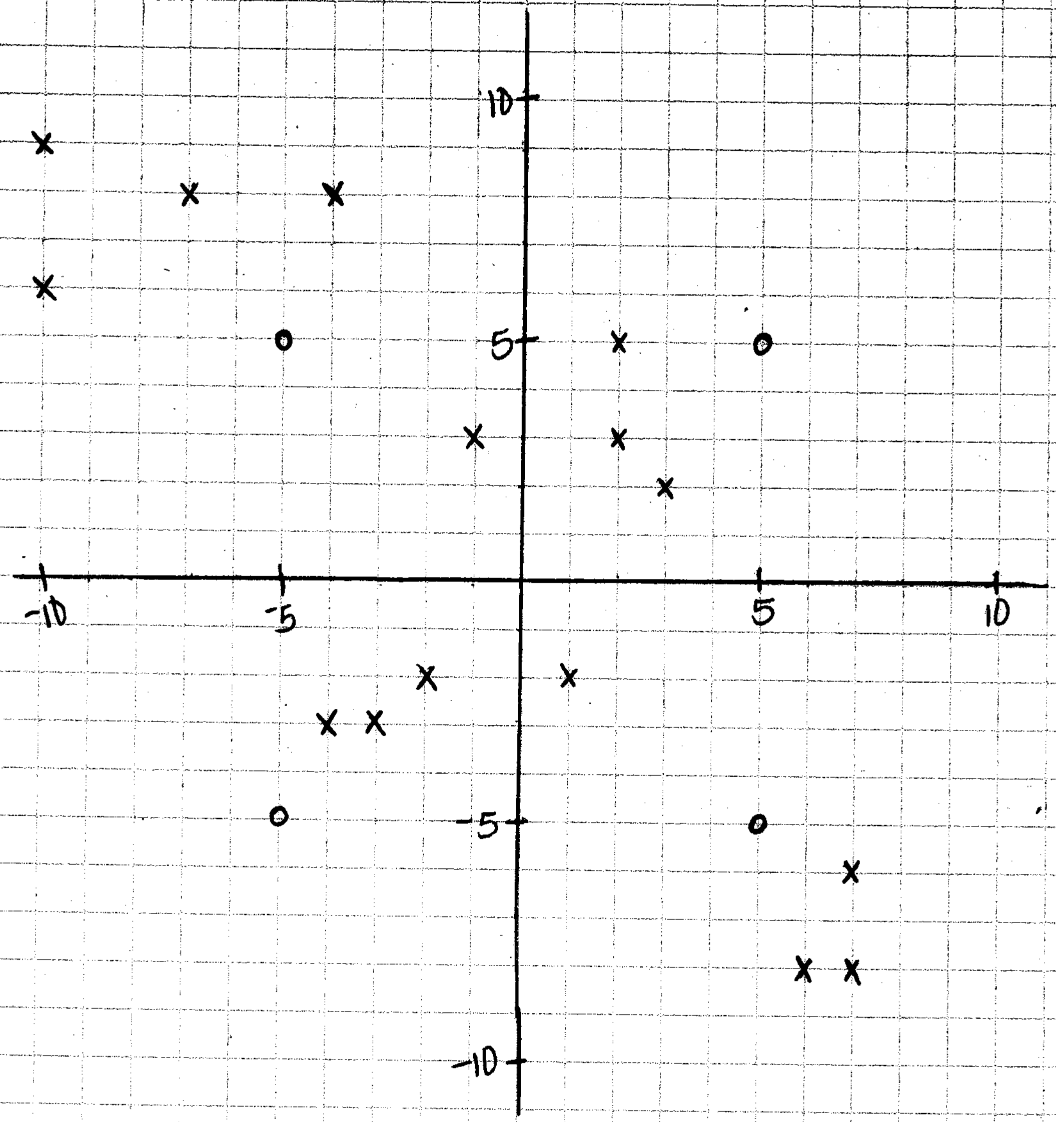
<u>NOT REQUESTED</u>

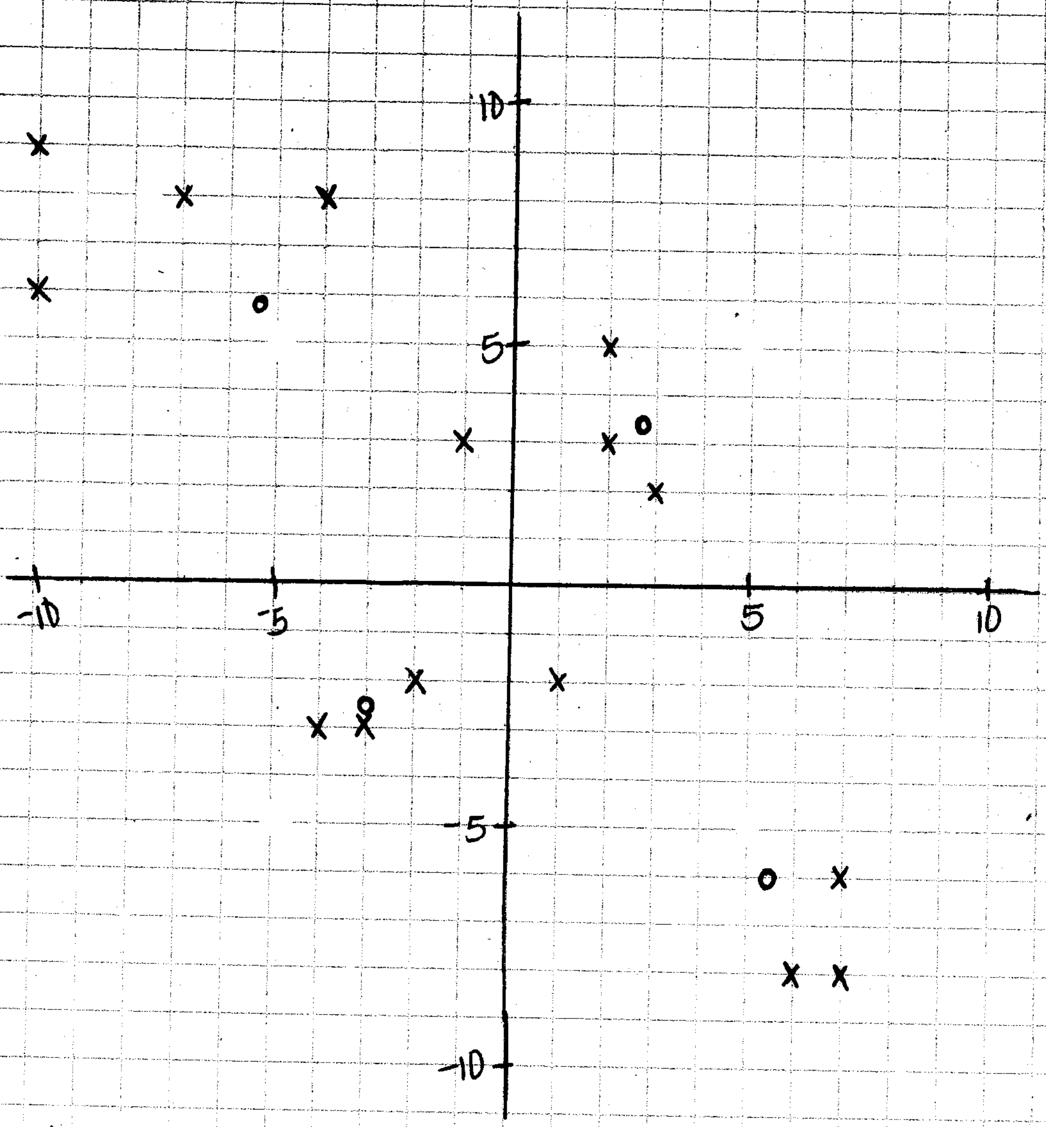
The Shannon lower bound for $r=1$ is

$$R=h(X)-\log 2eD=\log V-\log 2eD$$
$$=\log\frac{V}{2eD}\text{ or }D=\frac{V}{2e}2^{-R}.$$

Compare.

5.





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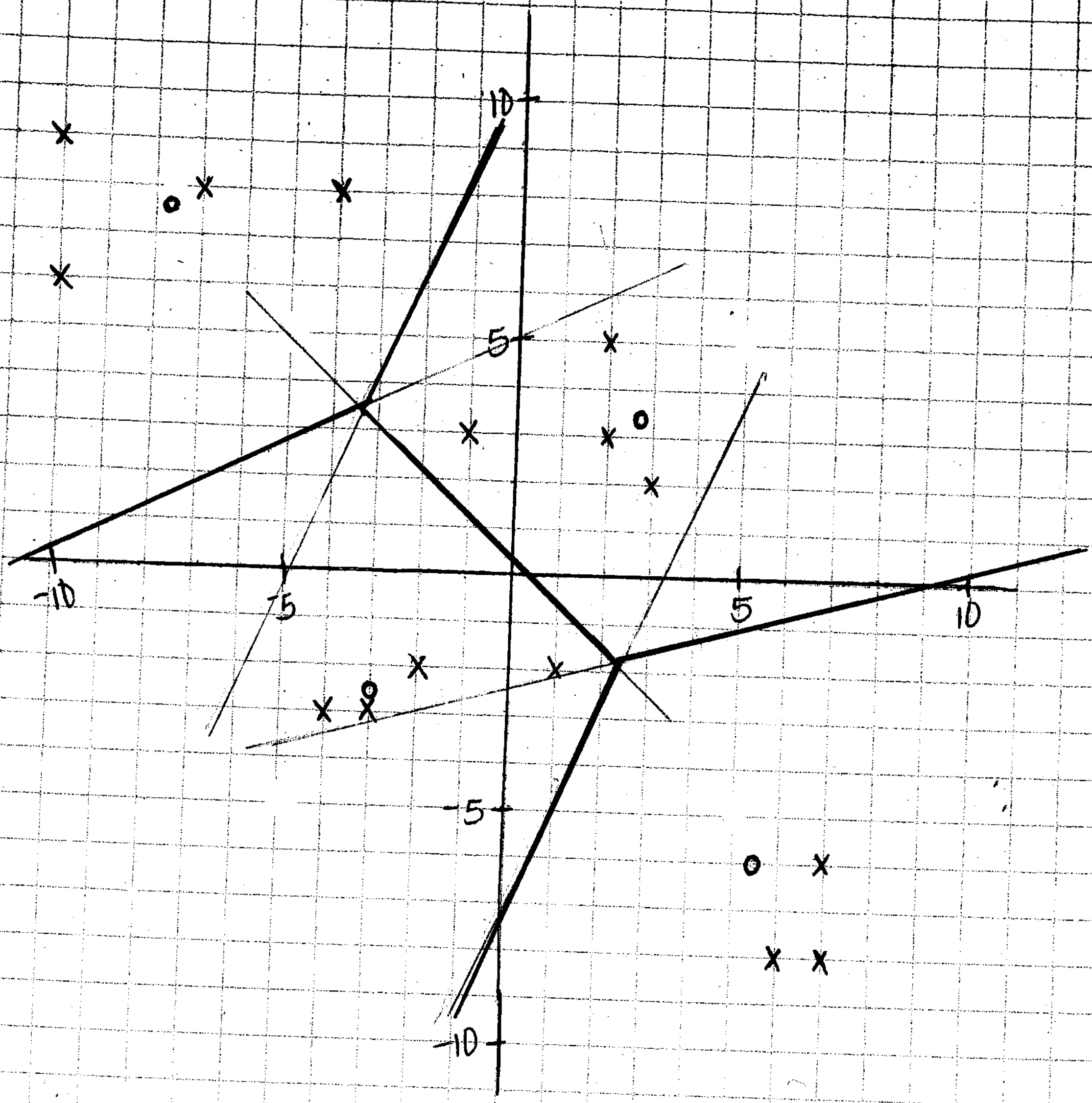
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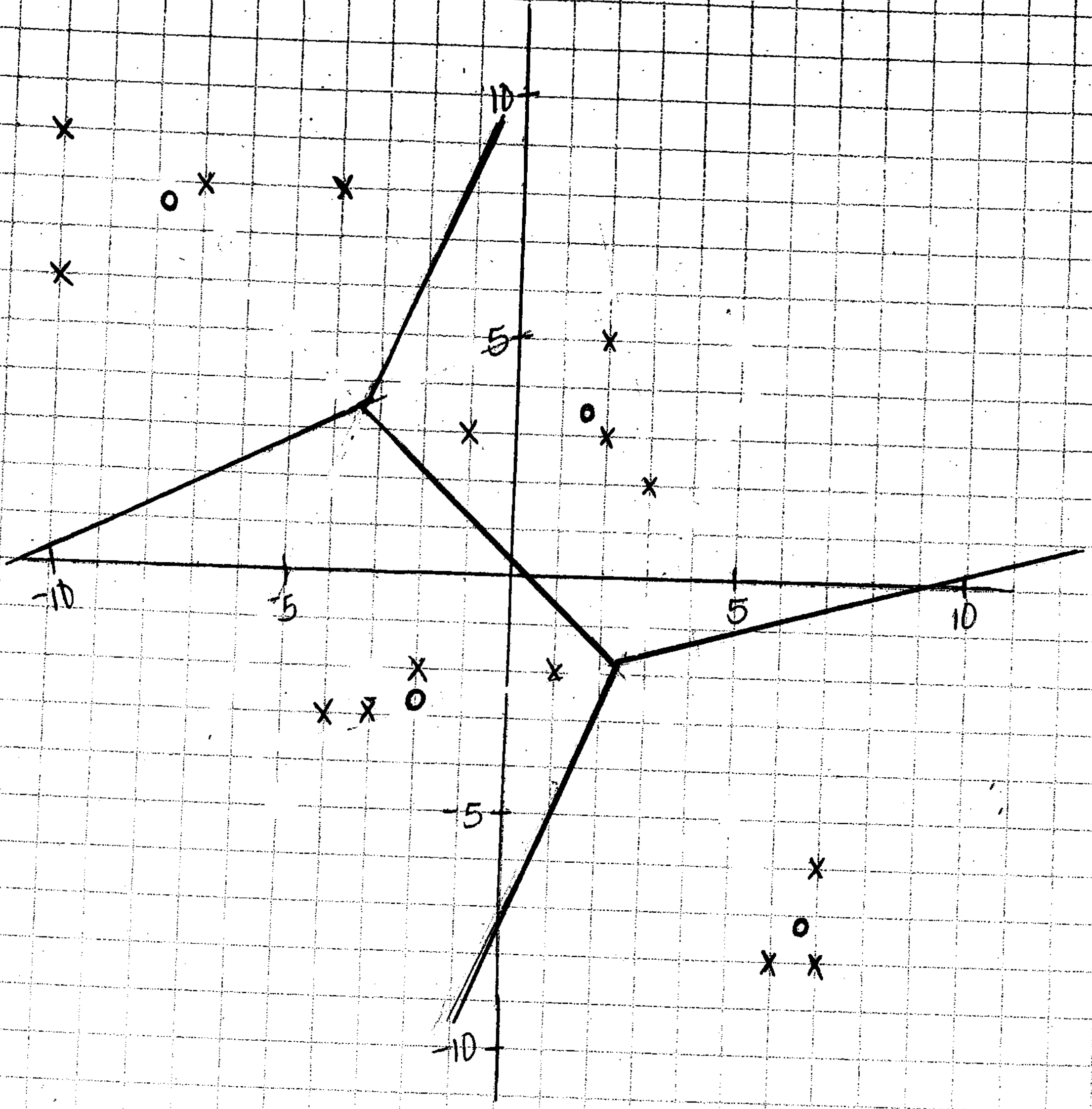

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