Homework No. 1 Solutions

1.) (a) Using characteristic functions,

$$\Phi_Z(\omega) = E\{e^{j\omega Z}\} = E\{e^{j\omega(X+Y)}\} =$$

$$\Phi_X(\omega)\Phi_Y(\omega), \text{ by independence}$$

$$= e^{j\omega\mu_X - \omega^2\sigma_X^2/2}\, e^{j\omega\mu_Y - \omega^2\sigma_Y^2/2}$$

$$= e^{j\omega(\mu_X + \mu_Y) - \omega^2(\sigma_X^2 + \sigma_Y^2)/2}$$

$\therefore$ Z is Gaussian with $\mu_Z = \mu_X + \mu_Y$ and $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.

(b) The pdf of the sum of two independent random variables is the convolution of their marginal densities. Hence,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx =$$

$$\int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\,dx = \lambda^2 e^{-\lambda z}\int_0^z dx$$

$$= \lambda^2 z e^{-\lambda z}, \quad 0 \le z < \infty.$$

2.) $H(X) = -0.7\log 0.7 - 0.3\log 0.3 - 0.3\log 0.3$

$= 1.18129\ldots$

$H(Y) = 3\left(-\frac{1}{3}\log 1/3\right) = \log_2 3 = 1.585$

3.) $\operatorname{var} X = \int_{-\Delta/2}^{\Delta/2} x^2\left(\frac{1}{\Delta}\right)dx = \frac{x^3}{3}\Big|_{-\Delta/2}^{\Delta/2} \cdot \frac{1}{\Delta} = \frac{\Delta^2}{12}$

4.) $S_X(f) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{0} e^{\tau} e^{-j2\pi f\tau} d\tau$

$+ \int_{0}^{\infty} e^{-\tau} e^{-j2\pi f\tau} d\tau = \frac{1}{1-j2\pi f}[1-0] + \frac{1}{-(1+j2\pi f)}[0-1]$

$= \frac{2}{1+4\pi^2 f^2}$

5.) For W_k statistically independent of past X_j, $R_X(j) = E X_k X_{k+j} = E[\alpha X_{k-1} + W_k][\alpha X_{k+j-1} + W_{k+j}]$

$$= \alpha^2 E X_{k-1} X_{k+j-1} + \alpha E X_{k-1} W_{k+j} + \alpha E X_{k+j-1} W_k + E W_k W_{k+j}$$

For $j > 0$, $R_X(j) = \alpha^2 R_X(j) + 0 + \alpha \alpha^{j-1} \sigma_W^2$

so $$R_X(j) = \frac{\alpha^j}{1-\alpha^2}\sigma_W^2$$

For $j = 0$, $R_X(0) = \alpha^2 R_X(0) + \sigma_W^2$

so $$R_X(0) = \frac{\sigma_W^2}{1-\alpha^2}.$$