Homework No. 2 Solutions

1. The necessary conditions for a minimum are obtained from

$$\frac{\partial D}{\partial y_j} = 0, \qquad \text{for all } j, \qquad \text{from which we find}$$

$$\frac{\partial D}{\partial y_j} = \frac{P_j}{\Delta_j}\int_{x_{j-1}}^{x_j}(-2)(x-y_j)dx = 0 \qquad \text{or} \qquad y_j = \frac{x_j + x_{j-1}}{2}$$

2.

$$P_U(u_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1.758} e^{-\gamma^2/2}\,d\gamma = 0.0392 = P_U(u_8)$$
$$P_U(u_2) = \frac{1}{\sqrt{2\pi}}\int_{-1.758}^{-1.172} e^{-\gamma^2/2}\,d\gamma = 0.0818 = P_U(u_7)$$
$$P_U(u_3) = \frac{1}{\sqrt{2\pi}}\int_{-1.172}^{-0.586} e^{-\gamma^2/2}\,d\gamma = 0.1556 = P_U(u_6).$$
$$P_U(u_4) = \frac{1}{\sqrt{2\pi}}\int_{-0.586}^{0} e^{-\gamma^2/2}\,d\gamma = 0.2224 = P_U(u_5).$$

The entropy of this DMS is

$$H(U) = 2.757 \text{ bits/letter.}$$

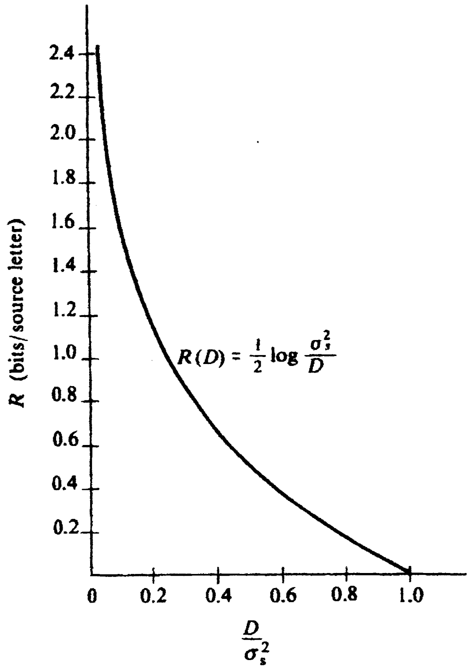
3.

$$P_U(u_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1.748} e^{-\gamma^2/2}\,d\gamma = 0.0401 = P_U(u_8)$$
$$P_U(u_2) = \frac{1}{\sqrt{2\pi}}\int_{-1.748}^{-1.050} e^{-\gamma^2/2}\,d\gamma = 0.1068 = P_U(u_7)$$
$$P_U(u_3) = \frac{1}{\sqrt{2\pi}}\int_{-1.050}^{-0.5006} e^{-\gamma^2/2}\,d\gamma = 0.1616 = P_U(u_6)$$
$$P_U(u_4) = \frac{1}{\sqrt{2\pi}}\int_{-0.5806}^{0} e^{-\gamma^2/2}\,d\gamma = 0.1915 = P_U(u_5).$$

The entropy is

$$H(U) = 2.824 \text{ bits/letter.}$$

4.


2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
R (bits/source letter)
$R(D) = \frac{1}{2}\log\frac{\sigma_s^2}{D}$
0
0.2
0.4
0.6
0.8
1.0
$\frac{D}{\sigma_s^2}$

5.| Problem 5.1

$$D = D_{gran} + D_{ol}$$

Uniform: $D_{gran} = \frac{\sigma^2}{3\beta^2 N^2}$ (5.4.4)

$$D_{ol} = \int_{-\infty}^{x_1} (x-y_1)^2 f_X(x)\,dx + \int_{x_{N-1}}^{\infty} (x-y_N)^2 f_X(x)\,dx$$

by symmetry

$$= 2\int_{x_{N-1}}^{\infty} (x-y_N)^2 f_X(x)\,dx$$

$$= 2\int_{V}^{\infty} \left(x-\left(V+\tfrac{\Delta}{2}\right)\right)^2 f_X(x)\,dx$$

$$\Delta = \frac{2\sigma}{N\beta} = \frac{2\sigma}{N}\,\frac{V}{\sigma} = \frac{2V}{N}$$

$$= 2\int_{V}^{\infty} \left(x - V\left(1+\tfrac{1}{N}\right)\right)^2 f_X(x)\,dx$$

$$= 2\int_{V}^{\infty} \left(x - \tfrac{\sigma}{\beta}\left(1+\tfrac{1}{N}\right)\right)^2 f_X(x)\,dx$$

with $f_X(x) = \frac{1}{\sqrt{2}\,\sigma} e^{-\frac{x\sqrt{2}}{\sigma}}$

$$= 2\int_V^\infty \left(x - \frac{\sigma}{\beta}\left(1+\frac{1}{N}\right)\right)^2 \frac{1}{\sqrt{2}\,\sigma} e^{-x\sqrt{2}/\sigma}\, dx$$

Let $z = \frac{\sqrt{2}}{\sigma}\left(x - \frac{\sigma}{\beta}\left(1+\frac{1}{N}\right)\right)$

$\therefore\ x = \frac{\sigma}{\sqrt{2}} z + \frac{\sigma}{\beta}\left(1+\frac{1}{N}\right)$; $dx = \frac{\sigma}{\sqrt{2}} dz$

so

$$D_{ol} = \left(1 + \frac{\sqrt{2}}{\beta N} + \frac{1}{\beta^2 N^2}\right)\sigma^2 e^{-\sqrt{2}/\beta}$$

and

$$SNR = 10 \log_{10}\left(\frac{\sigma^2}{D_{gran} + D_{ol}}\right).$$

as β varies.

μ-law $\quad G(x) = V \frac{\ln(1+\mu|x|/V)}{\ln(1+\mu)} \operatorname{sgn} x$

and from Class

$$D_{gran} = \frac{V^2}{3N^2}\int_{-V}^{V}\left(\frac{d}{dx}G(x)\right)^{-2} f_X(x)\, dx$$

D_{ol} as before.

Now
$$\frac{d}{dx} G(x) = \frac{1}{\ln(1+\mu)} \cdot \frac{\mu V}{V+\mu|x|}, \quad |x| \le V$$

$$\therefore D_{gran} = \frac{1}{12N^2}\int_{-V}^{V} \frac{4\ln^2(1+\mu)(V+\mu|x|)^2}{\mu^2}\cdot\frac{1}{\sigma\sqrt{2}}\, e^{-\sqrt{2}|x|/\sigma}\,dx$$

$$= \frac{\sqrt{2}}{3N^2\sigma}\ln^2(1+\mu)\int_0^V \left(\frac{V}{\mu}+x\right)^2 e^{-\sqrt{2}x/\sigma}\,dx$$

$$= \frac{\ln^2(1+\mu)}{3N^2}\cdot\frac{\sigma^2}{2}\int_0^V \left(\frac{\sqrt{2}V}{\mu\sigma}+\frac{\sqrt{2}x}{\sigma}\right)^2 e^{-\sqrt{2}x/\sigma}\,dx$$

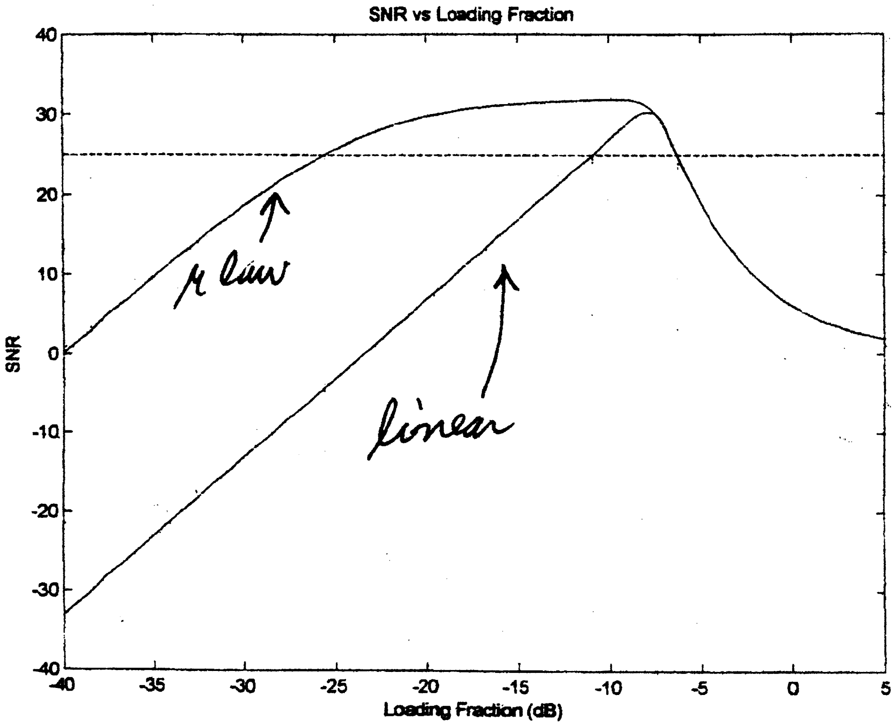
Making the change of variables,

$$y = \frac{\sqrt{2}V}{\mu\sigma}+\frac{\sqrt{2}x}{\sigma}, \quad dy = \frac{\sqrt{2}}{\sigma}$$

so

$$D_{gran} = \frac{\ln^2(1+\mu)}{3N^2}\cdot\frac{\sigma^2}{2}\int_{\frac{\sqrt{2}V}{\mu\sigma}}^{\frac{\sqrt{2}V}{\sigma}\left(\frac{1}{\mu}+1\right)} y^2 e^{-y}\,dy$$

$$= \frac{\ln^2(1+\mu)}{3N^2}\sigma^2\left[1+\frac{\sqrt{2}}{\mu\beta}+\frac{1}{\mu^2\beta^2}-\left(\frac{1}{\beta^2}\left(1+\frac{1}{\mu}\right)^2+\frac{\sqrt{2}}{\beta}\left(1+\frac{1}{\mu}\right)+1\right)e^{-\sqrt{2}/\beta}\right].$$


SNR vs Loading Fraction
SNR
Loading Fraction (dB)
40
30
20
10
0
-10
-20
-30
-40
-35
-25
-15
-5
5
μ law
linear