Signal Compression (ECE 242) **Feb 26, 2008**

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Homework 5

Solution

1. (Problem 8.1 on page 252 of the text book) Evaluate the high resolution quantization constant h of equation 8.2.3 for the case of a doubly exponential pdf.

   Solution:

$$h = \frac{1}{12}\left\{\int_{-\infty}^{\infty}[f_X(x)]^{\frac{1}{3}}dx\right\}^3.$$

$$f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|}.$$

$$\begin{aligned}
h &= \frac{1}{12}\left(\frac{\lambda^{\frac{1}{3}}}{2^{\frac{1}{3}}}\right)\left\{\int_{-\infty}^{\infty} e^{-\frac{\lambda}{3}|x|}dx\right\}^3 \\
&= \frac{\lambda}{24}\left\{\int_{-\infty}^{0} e^{\frac{\lambda x}{3}} + \int_{0}^{\infty} e^{\frac{-\lambda x}{3}}dx\right\}^3 \\
&= \frac{\lambda}{24}\left\{\frac{3}{\lambda}e^{\frac{\lambda x}{3}}|_{-\infty}^{0} - \frac{3}{\lambda}e^{-\frac{\lambda x}{3}}|_{0}^{\infty}\right\}^3 \\
&= \frac{\lambda}{24}\left\{\frac{3}{\lambda}(1) - \frac{3}{\lambda}(-1)\right\}^3 \\
&= \frac{\lambda}{24}\left(\frac{6}{\lambda}\right)^3 \\
&= \frac{9}{\lambda^2}.
\end{aligned}$$

   For Laplacian

$$\sigma^2 = E[X^2] = \frac{2}{\lambda^2} = 1.$$

Hence

$$\lambda = \sqrt{2}.$$

Therefore,

$$h = \frac{9}{\lambda^2} = \frac{9}{2}.$$

2. Problem 2: (Prob 8.3 on page 252 of text) Prove in two ways:

   (a) Lagrange Multiplier approach used in class

   (b) Arithmetic/Geometric Mean approach used in the text.

   Lagrange Multiplier Method:

   To prove: (8.37 for high resolution weighted average distortion case)

   We need to prove : The optimal bit assignments are given by

$$b_i = \bar{b} + \frac{1}{2}\log_2 \frac{\sigma_i^2}{\rho^2} + \frac{1}{2}\log_2 \frac{h_i}{H} + \frac{1}{2}\log_2 \frac{g_i}{G}$$

   Minimum overall distortion attained with this solution is

$$D = kHG\rho^2 2^{-2\bar{b}}$$

   We want to choose the b_i, $i = 1, 2, ..., k$ to minimize

$$J(\underline{b}) = D(\underline{b}) - \lambda(B - \sum_{i=1}^{k} b_i)$$

$$J(\underline{b}) = \sum_{i=1}^{k} g_i W_i(b_i) - \lambda(B - \sum_{i=1}^{k} b_i).$$

   Use the high resolution assumption

$$h_i = \frac{1}{12}\left\{\int_{-\infty}^{\infty} [f_X(x)]^{\frac{1}{3}}\, dx\right\}^3 \qquad (8.2.3)$$

$$D_i = \frac{\sigma_i^2}{12N^2}\left\{\int_{-\infty}^{\infty} \hat{f}_X(y)^{\frac{1}{3}} dy\right\}^3 = \frac{\sigma_i^2}{N^2} h_i$$

Also $N = 2^{b_i}$ gives

$$D_i = W_i(b_i) = h_i \sigma_i^2 2^{-2b_i}$$

This gives

$$J(\underline{b}) = \sum_{i=1}^{k} g_i h_i \sigma_i^2 2^{-2b_i} - \lambda \left(B - \sum_{i=1}^{k} b_i \right).$$

$$\frac{\partial J(\underline{b})}{\partial b_j} = 0$$

gives

$$-2 g_j h_j \sigma_j^2 2^{-2b_j} + \lambda = 0.$$

Solving for b_j, we get

$$b_j = \frac{1}{2}\log_2(g_j h_j \sigma_j^2) - \frac{1}{2}\log_2 \lambda.$$

This makes

$$B - \sum_{i=1}^{k} b_i = \sum_{i=1}^{k} \frac{1}{2}\log_2(g_j h_j \sigma_j^2) - k\frac{1}{2}\log_2 \lambda.$$

Therefore

$$-k\frac{1}{2}\log_2 \lambda = B - \sum_{i=1}^{k} \frac{1}{2}\log_2(g_j h_j \sigma_j^2).$$

Substituting back into b_j expression,

$$\begin{aligned} b_j &= \frac{1}{2}\log_2(g_j h_j \sigma_j^2) + \frac{1}{k}\left[B - \sum_{j=1}^{k}\frac{1}{2}\log_2(g_j h_j \sigma_j^2)\right] \\ &= \frac{1}{k}B + \frac{1}{2}\log_2(g_j h_j \sigma_j^2) - \frac{1}{2k}\sum_{j=1}^{k}\log_2(g_j h_j \sigma_j^2) \\ &= \bar{b} + \frac{1}{2}\left[\log_2(g_j h_j \sigma_j^2) - \frac{1}{k}\sum_{j=1}^{k}\log_2(g_j h_j \sigma_j^2)\right] \\ &= \bar{b} + \frac{1}{2}\log_2\left[\frac{(g_j h_j \sigma_j^2)}{\left(\prod_{j=1}^{k}(g_j h_j \sigma_j^2)\right)^{\frac{1}{k}}}\right] \end{aligned}$$

Substituting these values of b_j in $D(b)$, we get

$$\begin{aligned} D(b) &= \sum_{i=1}^{k} g_i h_i \sigma_i^2 2^{-2b_i} \\ &= \sum_{i=1}^{k} g_i h_i \sigma_i^2 2^{-2(\bar{b}+\frac{1}{2}\log_2\left[\frac{g_i h_i \sigma_i^2}{\left(\prod_{j=1}^{k} g_j h_j \sigma_j^2\right)^{\frac{1}{k}}}\right])} \qquad = \sum_{i=1}^{k} 2^{-2\bar{b}} G H \rho^2 \end{aligned}$$

Thus

$$D = kHG\rho^2 2^{-2b}.$$

SECOND PROOF: <u>Arithmetic Mean- Geometric Mean Inequality Approach.</u>

We want to prove that

(a) The optimal bit assignments are given by

$$b_i = \bar{b} + \frac{1}{2}\log_2\frac{\sigma_i^2}{\rho^2} + \frac{1}{2}\log_2\frac{b_i}{H} + \frac{1}{2}\log_2\frac{g_i}{G}$$

(b) Minimum overall distortion obtained with this solution is

$$D = kHG\rho^2 2^{-2\bar{b}}.$$

where g_i's are weight values, G is the geometric Mean of g_is, H is the geometric Mean of the h_i's.

If $N_i = 2^{b_i}, \quad i = 1, 2, 3, ..., k$ and $\sum_{i=1}^{k} \log N_i = B$ then

$$\sum_{i=1}^{k} h_i g_i \sigma_i^2 2^{-2b_i} \geq kHG\rho^2 2^{-2\bar{b}}$$

with equality if 8.3.6 is true.

$$\frac{1}{k}\sum_{i=1}^{k} h_i g_i \sigma_i^2 2^{-2b_i} \geq \left(\prod_{i=1}^{k} \sigma_i^2\right) HG2^{-2\bar{b}} \quad (AM - GMinequality) \quad (1)$$

This is because the geometric mean of $\{h_i g_i \sigma_i^2 2^{-2b_i}\}$ is

$$\left[\prod_{i=1}^{k} h_i g_i \sigma_i^2 2^{-2b_i}\right]^{\frac{1}{k}} = \left(\prod_{i=1}^{k} \sigma_i^2\right)^{\frac{1}{k}} HG2^{-2\bar{b}}.$$

Equality in equation [1] holds iff $h_i g_i \sigma_i^2 2^{-2b_i} = C$ a constant independent of i. From equation [1] the constant C must be $\rho^2 HG2^{-2\bar{b}}$. This makes

$$h_i g_i \sigma_i^2 2^{-2b_i} = \rho^2 HG2^{-2\bar{b}}.$$

We solve for b_i to get

$$\begin{aligned} -2b_i &= \log_2\left[\frac{\rho^2 HG2^{-2\bar{b}}}{h_i g_i \sigma_i^2}\right] \\ &= \log_2 2^{-2\bar{b}} + \log_2 \frac{\rho^2}{\sigma_i^2} + \log_2 \frac{H}{h_i} + \log_2 \frac{G}{g_i} \end{aligned}$$

Hence

$$b_i = -\frac{1}{2}(-2\bar{b}) - \frac{1}{2}\{\log_2 \frac{\rho^2}{\sigma_i^2} + \log_2 \frac{H}{h_i} + \log_2 \frac{G}{g_i}\}.$$

That is

$$b_i = \bar{b} + \frac{1}{2}\log_2 \frac{\sigma_i^2}{\rho^2} + \frac{1}{2}\log_2 \frac{b_i}{H} + \frac{1}{2}\log_2 \frac{g_i}{G}$$