ECE 242 Winter Quarter 2008
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Midterm Exam Solutions

1. Consider some estimate $\tilde{Y}$ and the corresponding error

$$\varepsilon^2(\tilde{Y}) = E[(Y-\tilde{Y})^t W(Y-\tilde{Y})] = E[(Y-\hat{Y}+\hat{Y}-\tilde{Y})^t W(Y-\hat{Y}+\hat{Y}-\tilde{Y})]$$
$$= E\left\|Y-\hat{Y}\right\|_W^2 + 2E[(Y-\hat{Y})^t W(\hat{Y}-\tilde{Y})] + E\left\|\hat{Y}-\tilde{Y}\right\|_W^2$$
$$\geq E\left\|Y-\hat{Y}\right\|_W^2 + 2E[(Y-\hat{Y})^t W(\hat{Y}-\tilde{Y})]$$

Working with the second term,
$E[(Y-\hat{Y})^t W(\hat{Y}-\tilde{Y})] = E_X E[(Y-\hat{Y})^t W(\hat{Y}-\tilde{Y}) \big| X]$
Given X, $\hat{Y}(X)$ and $\tilde{Y}(X)$ are deterministic functions so choose $\hat{Y}(X)$ such that $E[(Y-\hat{Y}(X)) \big| X] = 0 \Rightarrow \hat{Y}(X) = E[Y \big| X]$ so the conditional expectation above is zero and taking the expected value over the random variable X, the second term in the last inequality is zero and $\varepsilon^2(\tilde{Y}) \geq \varepsilon^2(\hat{Y})$.

2. (a) From class, $D \geq \frac{1}{12} e^{-2(H(Y)-h(X))}$ where $h(X) = \log_2 e\sqrt{2} = 1.943$ and $H(Y) = 3$, so $D \geq 0.01$ and $SNR \leq 20$ dB.
(b) $D(3) = \frac{e}{\pi} 2^{-6} = 0.0135$ or $SNR = 18.69$ dB.

3. (a) The output $SNR = 10\log_{10}\frac{\sigma^2}{D}$ where $D = \frac{x_{\max}^2}{3N^2}\int_{-x_{\max}}^{x_{\max}} f_X(x)\left(\frac{d}{dx}G(x)\right)^{-2} dx$ and the pdf is Laplacian so $f_X(x) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x|/\sigma}$. Substitute $\frac{d}{dx}G(x) = \frac{m}{1-e^{-m}} e^{-m|x|/x_{\max}}$ and evaluate.
(b) $\Delta = \frac{2x_{\max}}{N}\left(\frac{d}{dx}G(x)\right)^{-1}$ so find $\Delta_{\min}$ by letting $x \to 0$ and $\Delta_{\max}$ by letting $x \to x_{\max}$ so we get $\frac{\Delta_{\max}}{\Delta_{\min}} = e^m$
(c) The point density function $\lambda(x) = \frac{1}{2x_{\max}}\left(\frac{d}{dx}G(x)\right) = \frac{1}{2x_{\max}} \cdot \frac{m}{1-e^{-m}} e^{-m|x|/x_{\max}}$

4. (a) The loading factor is $\gamma = \frac{V}{\sigma}$, where the variance is 1 for the quantizers in the tables. I accepted $V = 4\Delta = 2.344$ or $V = 3\Delta = 1.758$

(b) High rate result: $SNR = 6.02r + 10\log_{10}(3/\gamma^2) = 18.06 - 2.63 = 15.37$ dB. Compare this to $SNR = 10\log_{10}\left(1/0.03744\right) = 14.26$ dB

(c) $D_{gran} = 2\sum_{i=1}^{4}\int_{(i-1)\Delta}^{i\Delta}[x - \frac{(2i-1)}{2}\Delta]^2 f_X(x)dx$ and $D_{ol} = 2\int_{4\Delta}^{\infty}[x - \frac{7\Delta}{2}\}^2 f_X(x)dx$ then substitute for the pdf.

5. (a) For the first quantizer, we input X, $\hat{X} = Q_1(X)$ and $E(X - \hat{X})^2 = EX^2 - E\hat{X}^2$. For the second quantizer, the input is $U = X - \hat{X}$, so $\hat{U} = Q_2(U)$ and we have

$E(U - \hat{U})^2 = EU^2 - E\hat{U}^2 = EX^2 - E\hat{X}^2 - E\hat{U}^2$. Overall,

$E(X - \tilde{X})^2 = E(X - \hat{X} - \hat{U})^2 = E(U - \hat{U})^2$

(b) For optimal MMSE Q_2, $E(U - \hat{U})\hat{U} = 0 \Rightarrow EU\hat{U} = E\hat{U}^2$