Homework 1

This homework covers **Lecture 2** (Probability) and **Lecture 3** (Random Processes). Although there are a large number of problems, the majority should take very little work on your part to answer – if you understand the concepts covered in class. Most problems are taken from Brown and Hwang [BH] or from Simon [Si] (see references). *WARNING: The final problem (which includes MATLAB programming) may take longer than the rest of the assignment, combined!*

P1.1 – a) BH.1.8, b) BH.1.9, c) BH.1.10, d) BH.1.25 (MATLAB's "nchoosek" function may help.)
(15 pts)

1.8 Assume equal likelihood for the birth of boys and girls. What is the probability that a four-child family chosen at random will have two boys and two girls, irrespective of the order of birth?

[*Note:* The answer is not $\frac{1}{2}$ as might be suspected at first glance.)

1.9 Consider a sequence of random binary digits, zeros and ones. Each digit may be thought of as an independent sample from a sample space containing two elements, each having a probability of $\frac{1}{2}$. For a six-digit sequence, what is the probability of having:

(a) Exactly 3 zeros and 3 ones arranged in any order?
(b) Exactly 4 zeros and 2 ones arranged in any order?
(c) Exactly 5 zeros and 1 one arranged in any order?
(d) Exactly 6 zeros?

1.10 A certain binary message is n bits in length. If the probability of making an error in the transmission of a single bit is p, and if the error probability does not depend on the outcome of any previous transmissions, show that the probability of occurrence of exactly k bit errors in a message is

$$P(k \text{ errors}) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{(P1.10)}$$

The quantity $\binom{n}{k}$ denotes the number of combinations of n things taken k at a time. (This is a generalization of Problems 1.8 and 1.9.)

1.25 Three similar "unfair" coins are tossed simultaneously. The coins are unfair in that $P(\text{Heads}) = .6$ and $P(\text{Tails}) = .4$. Let the discrete random variable X be defined to be the number of heads that result from the toss of the three coins. Find the discrete probability distribution associated with the random variable X.

Hint: As a check, total probability must sum to 1.

P1.2 – a) BH.1.12, b) BH.1.13, c) BH.1.20, d) BH.1.38.
(15 pts)

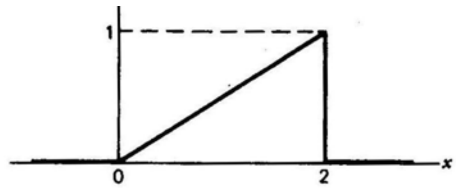
1.12 The random variable X may take on all values between 0 and 2, with all values within this range being equally likely.

(a) Sketch the probability density function for X.
(b) Sketch the cumulative probability distribution function for X.
(c) Calculate $E(X)$, $E(X^2)$, and Var X.

1.13 A random variable X has a probability density function as shown.

(a) Sketch the cumulative distribution function for X.

(b) What is the variance of X?



Problem 1.13

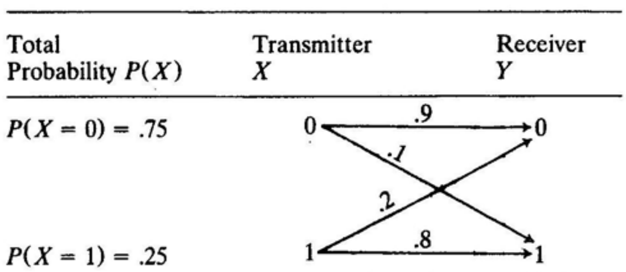
1.20 The diagram shown on the opposite page gives the error characteristics of a hypothetical binary transmission system. The numbers shown next to the arrows are the conditional probabilities of Y given X. The unconditional probabilities for X are shown to the left of the figure. Find:

(a) The conditional probabilities $P(X = 0|Y = 1)$ and $P(X = 0|Y = 0)$.

(b) The unconditional probabilities $P(Y = 0)$ and $P(Y = 1)$.

(c) The joint probability array for $P(X, Y)$.

Hint: Do part c first.



Problem 1.20

1.38 Two continuous random variables X and Y have a joint probability density function that is uniform inside the unit circle and zero outside, that is,

$$f_{XY}(x, y) = \begin{cases} 1/\pi, & (x^2 + y^2) \le 1 \\ 0, & (x^2 + y^2) > 1 \end{cases}$$

(a) Find the unconditional probability density function for the random variable Y and sketch the probability density as a function of Y.

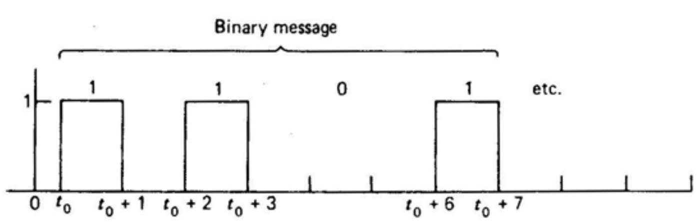
(b) Are the random variables X and Y statistically independent?

P1.3 – BH.2.3. Hint: For the PSD function, use Table A.2 on the last page of this homework.

(8 pts)

2.3 The waveform shown is an example of a digital-coded waveform. The signal is equally likely to be zero or one in the intervals $(t_0, t_0 + 1)$, $(t_0 + 2, t_0 + 3)$, etc., and it is always zero in the "in between" intervals $(t_0 + 1, t_0 + 2)$, $(t_0 + 3, t_0 + 4)$, etc. The switching time t_0 is random and uniformly distributed between zero and two. The presence or absence of a pulse in the "pulse possible" intervals is the code for a binary digit. There is no statistical correlation among any of the bits of the message. Find the autocorrelation and spectral density functions for this process.

Note: To "find" these results, provide two well-labeled plots. (Again, use Table A.2)



Problem 2.3

P1.4 – BH.1.15
(9 pts)

1.15 A random variable X whose probability density function is given by

$$f_X(x) = \begin{cases} \alpha\, e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is said to have an exponential probability density function. This density function is sometimes used to describe the failure of equipment components (12, 13). That is, the probability that a particular component will fail within time T is

$$P(\text{failure}) = \int_0^T f_X(x)\, dx \qquad \text{(P1.15)}$$

Note that α is a parameter that may be adjusted to fit the situation at hand.

(a) Find α for an electronic component whose average lifetime is 10,000 hours. ("Average" is used synonymously with "expectation" here.)

(b) Suppose we wish the probability of failure for the component of part (a) to be less than .01, that is, we wish the reliability to be .99. For what time span might we expect a reliability of .99?

(c) What is the *median* time to failure? (As opposed to the *mean*, in part a.)

P1.5 – BH.1.35.
(8 pts)

1.35 The vector Gaussian random variable

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is completely described by its mean and covariance matrix. In this case, they are

$$\mathbf{m}_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{C}_X = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

Now consider another vector random variable $\mathbf{Y}$ that is related to $\mathbf{X}$ by the equation

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the mean and covariance matrix for $\mathbf{Y}$.

P1.6 – a) Si.2.3, b) Si.2.5, c) Si.2.8.
(15 pts)

2.3 Determine the value of a in the function

$$f_X(x) = \begin{cases} ax(1-x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

so that $f_X(x)$ is a valid probability density function [Lie67].

2.5 The probability density function of an exponentially distributed random variable is defined as follows.

$$f_X(x) = \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $a \geq 0$.

a) Find the probability distribution function of an exponentially distributed random variable.
b) Find the mean of an exponentially distributed random variable.
c) Find the second moment of an exponentially distributed random variable.
d) Find the variance of an exponentially distributed random variable.
e) What is the probability that an exponentially distributed random variable takes on a value within one standard deviation of its mean?

2.8 Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w and σ_v, respectively. What is the standard deviation of the random variable $X = W + V$?

P1.7 – MATLAB exercise.
(30 pts)

a) Write a MATLAB program to solve the following problem. You are playing a casino game where the probability of winning is $p_W = 0.55$ and $p_L = 1 - p_W = 0.45$. At each game event, you bet one unit *if and only if* your current bankroll, n_k, is greater than zero. ***If you do bet,* then:**

Your total current bankroll goes up by one **if you win**: $n_{k+1} = n_k + 1$.
Your total bankroll goes down by one unit **if you lose**: $n_{k+1} = n_k - 1$.

Play continues forever – but you can only bet so long as $n > 0$. If you ever go "bankrupt" ($n = 0$), you remain bankrupt forever (regardless of future game outcomes). Begin with an initial bankroll of $N = 2$. The goal of your MATLAB m-file is to generate the following plots:

i) Plot the probability of being bankrupt ($n_k = 0$) after the kth game event:

$$P(n_k = 0 | N = 2, k)$$

(Note that this is a cumulative calculation, since going bankrupt at game event $k = k_{br}$ results in *remaining* bankrupt for all $k \geq k_{br}$.)

ii) Plot the cumulative expected winnings over time. (EV(n_k)-N, the current expect value of the bankroll minus the initial bankroll, as a function of k, the number of games that have occurred so far.

b) Consider a more general form of the game above: i.e., where you either win or lose each consecutive bet and where going bankrupt ends all future betting. Find a closed-form solution to the probability, $P(n_k = 0|N, k = \infty)$, of *ever* going bankrupt, given you begin with a bankroll of *N* bet units, and that the probability of winning at each step is p_w. (Hint: Imagine you start with *N=2*. The probability that you would eventually end up in a situation where *N=1* is identical to the probability that you would start at *N=1* and eventually end up bankrupt [*N=0*]. Try to write such a relationship algebraically to solve for P(1).)

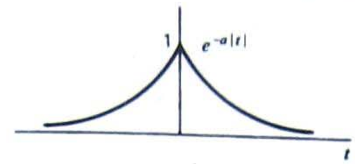
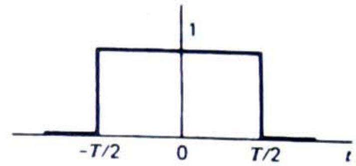
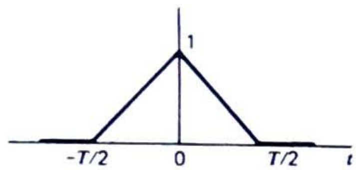
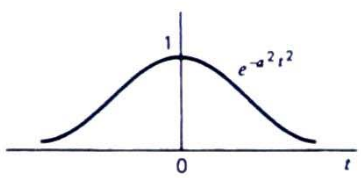
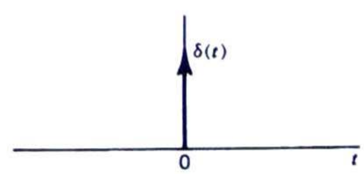
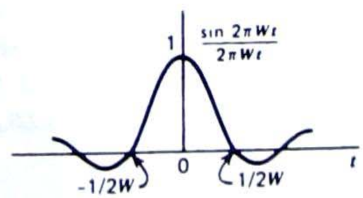
c) Write a second MATLAB program to solve the following revised betting problem. Now, instead of betting a constant amount (1 unit) at each step, you decide to bet a fraction of the current bankroll at each step. (Note we will assume we can bet any "fraction" of a monetary unit here, to avoid quantization issues in currency.) As long as you always bet a fraction, $f < 1$, of your bankroll at each step, you can never go bankrupt, of course. As in part a), assume $p_W = 0.55$ and $p_L = 1 - p_W = 0.45$. At each step, you will bet 10% ($f = 0.1$) of your current bankroll, B_k. Your program should provide the following outputs, for a game consisting of **exactly 100 game events**:
i) Plot the cumulative distribution function (CDF) of x: the ratio of the final bankroll to the initial bankroll. ($x = B_{100}/B_0$). (I suggest using semilogx() to plot this.)
ii) What is the expected value (mean)? (i.e., after the 100 games events occur…)
iii) What is the median?
iv) What is the mode?
v) What is the mean square value (MSV)?
vi) What is the variance?

References

[BH] Brown, R.G., and Hwang, P.Y.C. Introduction to Random Signals and Applied Kalman Filtering with MATLAB Exercises and Solutions, 3rd ed. Wiley, 1997.
[Si] Simon, D. Optimal State Estimation: Kalman, H-inf, and Nonlinear Approaches. Wiley, 2006.

Appendix

Table A.2. Common Fourier Transform Pairs

<table>
<tr><th>Name</th><th>Pictorial Description</th><th>Fourier Transform</th></tr>
<tr><td>Damped exponential</td><td>
</td><td>$\frac{2a}{\omega^2 + a^2}$</td></tr>
<tr><td>Rectangular pulse</td><td>
</td><td>$T\frac{\sin(\omega T/2)}{(\omega T/2)}$</td></tr>
<tr><td>Triangular pulse</td><td>
</td><td>$\frac{T}{2}\left[\frac{\sin(\omega T/4)}{(\omega T/4)}\right]^2$</td></tr>
<tr><td>Gaussian pulse</td><td>
</td><td>$\frac{\sqrt{\pi}}{a}e^{-(\omega^2/4a^2)}$</td></tr>
<tr><td>Symmetric impulse</td><td>
</td><td>1</td></tr>
<tr><td>Sinc function (sinc $2Wt$)</td><td>
</td><td>$F(j\omega) = \begin{cases} \frac{1}{2W}, & |\omega| < 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$</td></tr>
</table>