Homework 2

This homework covers **Lecture 4** (Linear system response to stochastic inputs) and **Lecture 5** (Least squares estimation).

P2.1 – BH.3.4. Steady-state response to a noise input.
(20 pts)

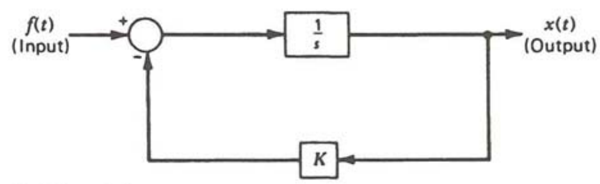
3.4 The input to the feedback system shown is a stationary Markov process with an autocorrelation function

$$R_f(\tau) = \sigma^2 e^{-\beta|\tau|}$$

The system is in a stationary condition.

(a) What is the spectral density function of the output?

(b) What is the mean-square value of the output?



Problem 3.4

P2.2 – BH.3.13. Transient response to a noise input. This will require a different technique than the last problem, since you are now to calculate the mean square value *as a function of time* (in case this is not clear in the problem statement).
(20 pts)

3.13 A certain linear system is known to satisfy the following differential equation:

$$\ddot{x} + a\dot{x} = f(t)$$

$$x(0) = \dot{x}(0) = 0$$

where $x(t)$ is the response and $f(t)$ is the input that is applied at $t = 0$. If $f(t)$ is white noise with spectral density amplitude A, what is the mean-square value of the response $x(t)$?

P2.3 – a) BH.3.6. b) BH.3.27. Parts (a) and (b) simply solve the same problem both analytically (in a) and numerically (in b) – so you can use part (b) to check your answer in (a). (Please note the "Additional Hint" at the end of this problem, too…)
(35 pts)

3.6 Find the steady-state mean-square value of the output for a first-order low-pass filter [i.e., $G(s) = 1/(1 + Ts)$] if the input has an autocorrelation function of the form

$$R(\tau) = \begin{cases} \sigma^2(1 - \beta|\tau|), & -\frac{1}{\beta} \le \tau \le \frac{1}{\beta} \\ 0, & |\tau| > \frac{1}{\beta} \end{cases}$$

[*Hint:* The input spectral function is irrational so the integrals given in Table 3.1 are of no help here. One approach is to write the integral expression for $E(X^2)$ in terms of real ω rather than s and then use conventional integral tables. Also, those familiar with residue theory will find that the integral can be evaluated by the method of residues.]

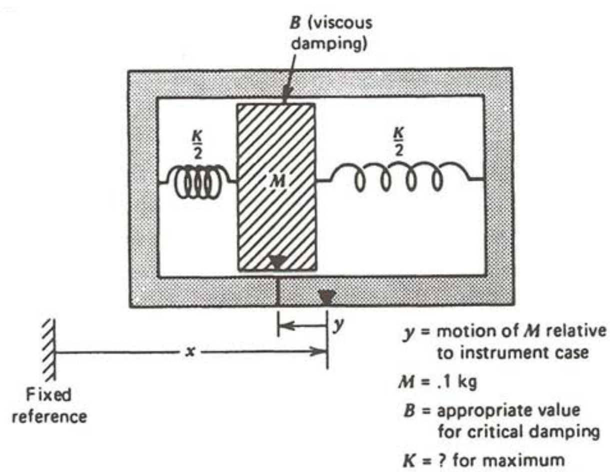
3.27 When it is difficult to integrate the power spectral density function in closed form, one should not overlook numerical integration. This is often the quickest way to get an answer to a specific numerical problem. In Problem 3.6, let $\sigma^2 = \beta = T = 1$, and repeat the problem using MATLAB's quad or quad8 numerical integration programs. Compare your result with the exact value of e^{-1}.
(*Note:* Beware of trying to integrate numerically through the origin where there is the indeterminate form 0/0. This can be avoided by staying an incremental distance away from the origin in the integration. Then approximate the omitted part as a narrow rectangular strip.)

Additional Hint: For BH.3.6, the hint given in the problem statement makes the integration somewhat easier – but it's still a bit messy. A second hint is that we know that the mean square value of the output will be the same for *any* input processes that have the same, specified autocorrelation, R_x. So an additional approach is simply to reason about $E(x^2)$ entirely within the "time domain" (without considering the PSD at all). That is, you can integrate to solve for $E(x^2)$ for the filter output given ***one particular input process*** that has the required autocorrelation. The trick is in finding a process that makes integration simple. (It may be helpful to PLOT what $R_x(\tau)$ actually looks like and to use your intuition from problem BH.2.3 on Homework 1…) Here are two example processes you might consider: (1) Imagine the input to G(s) is generated by taking sequential samples are drawn from a Gaussian distribution, with each sample being "held" for some particular time before the next sample is drawn. Or, (2) Imagine the input to G(s) is a sequence of pulses that alternate in length, where a short pulse (lasting for a period T_S) that has a 50/50 chance of being $+K$ or $-K$ (i.e., positive or negative, but of the same amplitude) is followed by a much longer pulse (with period $T_L >> T_S$) that is equal to zero. Here, T_L is chosen to be "long enough" that any the effects of past pulses can be ignored (in the limit). If you take this "time domain" approach, make sure that whatever process you integrate is scaled such that it has the required autocorrelation. For some processes, you should find that the resulting integrals are relatively simple to compute by hand (i.e., no table necessary). Good luck! (Please be sure to check that that BH.3.27 and BH.3.6 do in fact agree for different values of the variables.)

P2.4 – (a) BH.4.3. This problem – as written – is a bit annoying, because the answer (for this over-idealized model) turns out to be "make K as large as possible". So (given this hint), instead just calculate the corresponding power. (b) Now, calculate the natural frequency, $\omega_n = \sqrt{K/M}$, that will yield 96% of the optimum power output. [Hint: How is this frequency related to the autocorrelation given? e.g., how does it relate to the variable β in the PSD function for $R_{\dot{x}}(\tau)$?] (c) What natural frequency would yield 75% of the optimal power possible?
(25 pts)

4.3 One facet of biomedical electronic instrumentation work is that of implanting miniature, telemetering transducers in live animals. This is done in order that various body functions may be observed under normal conditions of activity and environment. When considering such an implant, one is immediately confronted with the problem of supplying energy to the transducer. Batteries are an obvious possibility, but they are bulky and have finite life. Another possibility is to take advantage of the random motion of the animal, and a device for accomplishing this is shown in the simplified diagram of the accompanying figure. The energy conversion takes place in the damping mechanism. This is shown as simple viscous damping, but, in fact, would have to be some sort of electromechanical conversion device. Assuming that all the power indicated as being dissipated in damping can be converted to electrical form, what is the optimum value for the spring constant K and how much power is converted for this optimum condition? The spring–mass arrrangement is to be critically damped, the mass is .1 kg, and the autocorrelation function for the velocity $\dot{x}$ is estimated to be

$$\boldsymbol{R_{\dot{x}}(\tau) = \sigma^2 e^{-\beta|\tau|} = 1e^{-2\pi|\tau|}}\textbf{(ft/sec)}^2$$



Problem 4.3

Hints: I suggest you begin by determining the transfer function from the absolute velocity of the case, dx/dt (sX(s) in the Laplace domain), to the relative velocity of the mass with respect to the case, dy/dt (sY(s)). Also, note that power for this mechanical system can be written as a velocity times the force due to damping, which can be written in a manner analogous to the familiar "$P = I^2R$" representation of power (current times voltage) for an electrical system – so power can be expressed in terms of $E(\dot{y}^2)$ (i.e., mean square value of the output), which we can find via spectral factorization. Finally, solving things in terms of both β of the autocorrelation and ω_n of the spring-mass-damping system may be helpful and yield more intuition.