ECE 277B Fall 2007 HW#3 Solutions

Solution #1 (2.16)

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta} = \frac{\partial p(\mathbf{x};\theta)}{\partial\theta}\frac{1}{p(\mathbf{x};\theta)}$$

By the definition of the mean

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty}\frac{\partial p(\mathbf{x};\theta)}{\partial\theta}\frac{p(\mathbf{x};\theta)}{p(\mathbf{x};\theta)}d\mathbf{x} = \int_{-\infty}^{\infty}\frac{\partial p(\mathbf{x};\theta)}{\partial\theta}d\mathbf{x}\\ &\overset{(a)}{=} \frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}p(\mathbf{x};\theta)d\mathbf{x} = \frac{\partial 1}{\partial\theta} = 0\end{aligned}$$

Interchanging order of differentiation and integration in (a) is allowed since the limits do not depend on θ.

Solution #2

2.17 The likelihood function is

$$P(X;q) = \prod_{i=1}^{N} q^{x_i}(1-q)^{(1-x_i)}$$

or

$$P(X;q) = q^{\sum_{i=1}^N x_i}(1-q)^{(N-\sum_{i=1}^N x_i)}$$

$$\begin{aligned}\frac{\partial P(X;q)}{\partial q} &= (\sum_{i=1}^{N} x_i)q^{(\sum_{i=1}^N x_i - 1)}(1-q)^{(N-\sum_{i=1}^N x_i)}\\ &-(N-\sum_{i=1}^{N}x_i)(1-q)^{(N-\sum_{i=1}^N x_i - 1)}q^{\sum_{i=1}^N x_i} = 0\end{aligned}$$

or

$$q^{\sum_{i=1}^N x_i}(1-q)^{(N-\sum_{i=1}^N x_i)}\left(\frac{\sum_{i=1}^N x_i}{q} - \frac{N-\sum_{i=1}^N x_i}{1-q}\right) = 0$$

The solutions $q = 0, 1$ result in a minimum of $P(X;q)$. The maximum comes from

$$\frac{\sum_{i=1}^N x_i}{q} - \frac{N-\sum_{i=1}^N x_i}{1-q} = 0 \Rightarrow q = \frac{1}{N}\sum_{i=1}^{N}x_i$$

Solution #3

2.20 The likelihood function is

$$\begin{aligned}L(\theta) &= \sum_{k=1}^{N}\ln\theta^2 + \sum_{k=1}^{N}\ln x_k - \sum_{k=1}^{N}\theta x_k\\ &= N\ln\theta^2 + \sum_{k=1}^{N}\ln x_k - \theta\sum_{k=1}^{N}x_k\end{aligned}$$

or

$$\frac{\partial L(\theta)}{\partial\theta} = \frac{2N\theta}{\theta^2} - \sum_{k=1}^{N}x_k = 0$$

and finally

$$\hat{\theta}_{ML} = \frac{2N}{\sum_{k=1}^N x_k}$$

Solution #4

2.23 From the theory, we have that the MAP estimate corresponds to the maximum of

$$Q(\mu) \equiv \sum_{k=1}^{N} \ln p(x_k; \mu) + \ln p(\mu)$$

$$= -\sum_{k=1}^{N} \frac{(x_k - \mu)^2}{2\sigma^2} + \ln \mu - \frac{\mu^2}{2\sigma_\mu^2} - \ln \sigma_\mu^2$$

or

$$\frac{\partial Q(\mu)}{\partial \mu} = -\sum_{k=1}^{N} \frac{\mu - x_k}{\sigma^2} + \frac{1}{\mu} - \frac{\mu}{\sigma_\mu^2} = 0$$

or

$$(\frac{N}{\sigma^2} + \frac{1}{\sigma_\mu^2})\mu^2 - Z\mu - 1 = 0$$

and finally

$$\mu = \frac{Z}{2R}(1 + \sqrt{1 + \frac{4R}{Z^2}})$$

since $\mu > 0$.

Solution #5

2.24

$$L(\theta) = -\sum_{k=1}^{N} \frac{(\ln x_k - \theta)^2}{2\sigma^2} - \sum_{k=1} N \ln(\sigma\sqrt{2\pi} x_k)$$

$$\frac{\partial L(\mu)}{\partial \theta} = 0 = -\sum_{k=1}^{N} \frac{\theta - \ln x_k}{\sigma^2} \Rightarrow \hat{\theta}_{ML} = \frac{1}{N}\sum_{k=1}^{N} \ln x_k$$