Fourier Analysis for Engineers

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Chapter 1
Basic real analysis

A good reference for this chapter is the book *Advanced Mathematical Analysis* by Richard Beals.

We will denote the set of real numbers by \( \mathbb{R} \). We will assume that the basic properties of the real numbers are obvious. The one non-obvious property we will assume as an axiom.

**Definition 1.1.** A number \( M \) is said to be an **upper bound** for a subset \( S \) of \( \mathbb{R} \) if \( s \leq M \) for all \( s \in S \).

**Axiom 1.2.** If \( S \) is a subset of \( \mathbb{R} \) with a finite upper bound, then there exists a number \( \text{lub}(S) \), called the **least upper bound** of \( S \), which is an upper bound of \( S \) that is less than every other upper bound of \( S \).

**Exercise 1.1.** Show that for every \( \epsilon > 0 \) there is an \( x \in S \) such that \( x > \text{lub}(S) - \epsilon \).

**Exercise 1.2.** Give an example of a bounded subset of the rational numbers that has no greatest lower bound.

**Definition 1.3.** A number \( L \) is said to be a **lower bound** for a subset \( S \) of \( \mathbb{R} \) if \( s \geq L \) for all \( s \in S \).

**Lemma 1.4.** If \( S \) is a subset of \( \mathbb{R} \) with a finite lower bound, then there exists a number \( \text{glb}(S) \), called the **greatest lower bound** of \( S \), which is a lower bound of \( S \) that is greater than every other lower bound of \( S \).

**Proof.** Exercise. **Hint:** Apply axiom 1.2 to \(-S\). \( \square \)

**Notation 1.5.** The least upper bound of \( S \) is also called the **supremum** of \( S \) and denoted by \( \sup(S) \). Similarly the greatest lower bound of \( S \) is also called the **infimum** of \( S \) and denoted by \( \inf(S) \).

**Exercise 1.3.** Show that \( \inf(S) \leq \sup(S) \).

**Definition 1.6.** Let \( a < b \). Then the subset \( \{ x \mid a < x < b \} \) of \( \mathbb{R} \) is called an **open interval** and denoted by \((a, b)\).

**Definition 1.7.** An **open ball** centered at \( a \) of radius \( r \) is the open interval \((a - r, a + r)\) and is denoted by \( B_r(a) \).

**Definition 1.8.** A **neighborhood** of a point is an open ball centered at that point.

**Definition 1.9.** A subset \( S \) of \( \mathbb{R} \) is said to be **open** if every point in \( S \) has a neighborhood that is contained in \( S \). The empty set is assumed to be open.

**Exercise 1.4.** Show that an open interval is an open set.

**Proposition 1.10.** The (possibly infinite) union of open sets is open.

**Proof.** Exercise. \( \square \)

**Proposition 1.11.** The finite intersection of open sets is open.

**Proof.** Exercise. \( \square \)
**Definition 1.12.** A point $x$ is said to be a limit point of a subset $S$ if every neighborhood of $x$ also contains a point of $S$ other than $x$ itself.

**Exercise 1.5.** Show that $a$ and $b$ are limit points of the open interval $(a,b)$.

**Definition 1.13.** Let $a \leq b$. Then the subset $\{x \mid a \leq x \leq b\}$ of $\mathbb{R}$ is called a closed interval and denoted by $[a,b]$.

**Definition 1.14.** A subset of $\mathbb{R}$ is said to be closed if it contains all its limit points. The empty set is assumed to be closed.

**Exercise 1.6.** Let $-\infty < a < b < \infty$. Show that $[a,b]$ is a closed set.

**Exercise 1.7.** Show that a finite subset of $\mathbb{R}$ has no limit points. Thus they are closed (vacuously).

**Exercise 1.8.** Show that if $x \notin S$ is not a limit point of $S$ then there is a neighborhood of $x$ that does not intersect $S$.

**Exercise 1.9.** Show that the finite union of closed sets is a closed set. **Hint:** Show that if $x$ is not in the union then it has a ball, one for each closed set, that does not intersect that closed set. The intersection of these balls gives a non-intersecting ball. Thus the union is not missing a limit point.

**Exercise 1.10.** Show that the (possibly infinite) intersection of closed sets is a closed set.

**Notation 1.15.** If $S$ is a subset let $S^c = \{x \mid x \notin S\}$ denote the complement of $S$.

**Exercise 1.11.** Show that $(\bigcup_n S_n)^c = \bigcap_n S_n^c$, where $S_n$ is a set. The number of sets can be infinite.

**Exercise 1.12.** Show that an open set can be written as the (possibly infinite) union of open intervals.

**Proposition 1.16.** A subset is open iff its complement is closed.

**Proof.** Exercise. \qed

**Definition 1.17.** A sequence of numbers $x_n$ for $n = 1, 2, \ldots$, is said to have a limit $x$, if for every $\epsilon > 0$ there exists an $N$ such that for all $n > N$, $|x_n - x| < \epsilon$. We denote this by $\lim_{n \to \infty} x_n = x$, and say that the sequence $x_n$ converges to $x$. If such an $x$ does not exist we say that the sequence $x_n$ diverges.

**Exercise 1.13.** Show that the sequence $x_n = n$ for $n = 1, 2, \ldots$, diverges.

**Exercise 1.14.** Show that the sequence $x_n = (-1)^n$ for $n = 1, 2, \ldots$, diverges.

**Exercise 1.15.** Show that the limit of a sequence is unique if it exists.

**Exercise 1.16.** Show that a convergent sequence is bounded.

**Exercise 1.17.** Let $x_n$ be a bounded sequence of non-decreasing real numbers; that is, there is an $M < \infty$ such that $|x_n| < M$, and $x_n \leq x_{n+1}$. Show that $\lim_{n \to \infty} x_n$ exists. **Hint:** Look at the supremum of the set $\{x_n\}$.

**Exercise 1.18.** Show that if $a > 0$ then $\lim_{n \to \infty} a^{1/n} = 1$. **Hint:** If $a < 1$ then this is an increasing sequence.

**Exercise 1.19.** Show that $\lim_{n \to \infty} n^{1/n} = 1$. **Hint:** Let $n^{1/n} = 1 + x_n$. Then by the binomial expansion $n \geq 1 + n x_n + \frac{1}{2} n(n-1) x_n^2 \geq \frac{1}{2} n(n-1) x_n^2$.

Let $x_n$ be a bounded real sequence and define $A_N = \{x_n \mid n \geq N\}$, $l_N = \inf(A_N)$ and $u_N = \sup(A_N)$.

**Exercise 1.20.** Show that $l_N$ is a bounded non-decreasing sequence and $u_N$ is a bounded non-increasing sequence. Therefore both $\lim_{n \to \infty} l_N$ and $\lim_{n \to \infty} u_N$ exist.

**Exercise 1.21.** Let $x_1 = -1$, $x_2 = 4$ and $x_n = n \mod 4$ for $n = 3, 4, \ldots$. Find $\lim_{n \to \infty} l_N$ and $\lim_{n \to \infty} u_N$ for this sequence.

**Definition 1.18.** Let $x_n$ be a bounded real sequence. We call $\lim_{n \to \infty} l_N$ as the lower limit of the sequence $x_n$ and denote it by $\liminf x_n$. We call $\lim_{n \to \infty} u_N$ as the upper limit of the sequence $x_n$ and denote it by $\limsup x_n$.

**Exercise 1.22.** Show that $l_N \leq u_M$.

**Exercise 1.23.** Show that $\liminf x_n \leq \limsup x_n$. 

**Exercise 1.24.** Show that if \( \lim \inf x_n < \lim \sup x_n \), then the sequence \( x_n \) diverges.

**Exercise 1.25.** Show that \( \lim_{n \to \infty} x_n = \lim \inf x_n = \lim \sup x_n \) iff the sequence is converging.

**Definition 1.19.** A sequence of numbers \( x_n \) for \( n = 1, 2, \ldots \), is said to be a **Cauchy sequence** if for every \( \epsilon > 0 \), there is an integer \( N \) such that for all \( n, m > N \), \( |x_n - x_m| < \epsilon \).

**Exercise 1.26.** Show that if a sequence of real numbers converges then it is a Cauchy sequence.

**Exercise 1.27.** Show that a Cauchy sequence is bounded.

**Exercise 1.28.** Show that for a Cauchy sequence \( \lim \inf S_n = \lim \sup S_n \). Otherwise we say that the infinite series diverges.

While this definition seems obvious and well-defined, it is not the case. The following two examples are from Beals.

**Example 1.20.** The series \( \sum_{n=1}^{\infty} n^{-1} \) diverges. To see this observe that

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
\geq \frac{1}{2} + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \cdots \\
= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
= \infty.
\]

**Exercise 1.29.** Show that if \(|x| < 1\) then \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \).

**Exercise 1.30.** **Riemann.** Show that the sum \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) can be rearranged to converge to any real number.

This is the reason why convergence as we have defined it for series can be problematic.

**Example 1.21.** The series \( \sum_{n=1}^{\infty} n^{-2} \) converges. To see this observe that

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{4} \right)^2 + \left( \frac{1}{5} \right)^2 + \left( \frac{1}{6} \right)^2 + \left( \frac{1}{7} \right)^2 + \left( \frac{1}{8} \right)^2 + \cdots \\
\leq 1 + \frac{1}{2^2} + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{4} \right)^2 + \left( \frac{1}{4} \right)^2 + \left( \frac{1}{4} \right)^2 + \cdots \\
= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\
= 2.
\]

**Exercise 1.31.** Show that if the sum \( \sum_n x_n \) converges then \( \lim_{n \to \infty} x_n = 0 \). **Hint:** The partial sums are a Cauchy sequence. Consider \( |S_n - S_{n+1}| \).

**Exercise 1.32.** Generalize the above two examples to show that if \( 0 \leq a_{n+1} \leq a_n \) then \( \sum a_n \) converges iff \( \sum 2^k a_{2^k} \) converges.

**Exercise 1.33.** Show that \( \sum n^{-1} \log n \) converges while \( \sum n^{-1} (\log n)^{-1} \) diverges.

**Exercise 1.34.** Let \( 0 \leq a_{n+1} \leq a_n \) and let \( f \) be a monotonic non-increasing function such that \( f(n) = a_n \). Show that \( \sum a_n \) converges iff \( \int_{1}^{\infty} f(x) \, dx \) converges.

**Exercise 1.35.** Show that \( \sum a_n \) converges for \( a_n = n^{-p} \) if \( p > 1 \) and diverges if \( p \leq 1 \).

**Proposition 1.22.** **Comparison test.** Let \( \sum a_n \) be a positive converging series. Then if \( |x_n| \leq a_n \), then \( \sum x_n \) is also converging.
Definition 1.24. Let \( S \) be a subset of \( \mathbb{R} \). A family \( A_n \) of sets is said to cover \( S \) if \( S \subset \bigcup_n A_n \).

Definition 1.25. A subset \( S \) of \( \mathbb{R} \) is said to be compact if every open cover of \( S \) has a finite sub-cover. That is, if the family of open sets \( A_n \) covers \( S \), then there are sets \( A_{n_1}, A_{n_2}, \ldots, A_{n_m} \) such that \( S \subset \bigcup_{k=1}^m A_{n_k} \).

Example 1.26. \( \mathbb{R} \) is not compact since the family \( A_n = (n, n+2) \) has no finite sub-cover.

Example 1.27. The set \((0, 1)\) is not compact. The open cover \( A_n = \left( \frac{1}{n}, 1 \right) \) has no finite sub-cover.

Example 1.28. A finite subset of \( \mathbb{R} \) is compact.

Theorem 1.29. Bolzano-Weierstrass. Let \( x_n \) be a sequence in a closed and bounded subset \( S \) of \( \mathbb{R} \). Then there is a sub-sequence of \( x_n \) that converges to a point in \( S \).

Proof. If there are only finite number of distinct points in \( x_n \) there is nothing to prove. So assume otherwise. Since \( S \) is bounded we can find a closed interval \( I_0 = [a_0, b_0] \) that covers \( S \). Since there are infinitely many distinct points in \( x_n \) one of the two intervals \([a_0, \frac{a_0 + b_0}{2}]\) and \([\frac{a_0 + b_0}{2}, b_0]\) must contain infinitely many points of \( x_n \). Call that interval \( I_1 = [a_1, b_1] \). Pick a point of the sequence \( x_{n_1} \) in \( I_1 \) and remove it from further consideration. Proceed thus to construct \( I_k = [a_k, b_k] \) and a corresponding point of the sequence \( x_{n_k} \) in \( I_k \). Note that \( \lim b_k - a_k = 0 \), since the intervals are halved every time and we started with an interval of finite length. Clearly the sequence we constructed is Cauchy. Exercise: finish the proof.

Theorem 1.30. Heine-Borel theorem. A subset of \( \mathbb{R} \) is compact iff it is closed and bounded.

Proof. Suppose \( S \) is a compact subset. For each \( x \in S \) let \( A_x = B_1(x) \). Since this is an open cover of \( S \) there is a finite sub-cover from it. From this it follows that \( S \) is bounded. Exercise: provide the details.

Let \( x \in S^c \). For every \( y \in S \) there is a ball centered at \( y \) that does not include \( x \). These provide an open cover of \( S \). There is a finite sub-cover from these balls none of which intersect \( x \). Hence there is a ball at \( x \) that does not intersect the finite sub-cover, and hence does not intersect \( S \). Therefore \( S^c \) is open.

To prove the converse let \( S \) be a closed and bounded subset of \( \mathbb{R} \). Let \( A_n \) be an open cover of \( S \). For each \( x \in A_n \) take every ball centered at \( x \) that is entirely contained in \( A_n \). The set of all such balls forms an open cover of \( S \). Clearly if we can prove that there is a finite sub-cover from this set we can get a corresponding finite sub-cover for \( A_n \). So we restrict our attention to these open balls at each \( x \in S \). For each \( x \in S \) consider the supremum of the radii of all balls centered at \( x \) that are in the open cover and pick a ball whose radius is at least half this supremum. Call this ball at \( x \) as \( B_{r(x)}(x) \). We claim that the infimum of \( r(x) \) is strictly greater than \( 0 \). Suppose it is not. Then the infimum is \( 0 \). This implies that there is a sequence in \( x_n \) with \( \lim_{n \to \infty} r(x_n) = 0 \). Pick a converging sub-sequence of \( x_n \) and call that as \( x_n \). Let \( x = \lim_{n \to \infty} x_n \in S \). But there is a finite open ball at \( x \) in the open cover. Which implies that for every \( x_n \) in this ball there is a ball of sufficiently large radius. Therefore the infimum cannot be \( 0 \) since we tried to pick large enough balls. Therefore for every \( x \in S \) we can pick a ball in the open cover whose radius is bigger than some \( \delta > 0 \). Exercise: from this show that we can pick a finite number of open balls to cover \( S \) since it is bounded.
Example 1.31. The set \( \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\} \) is compact since it is closed and bounded.

**Definition 1.32.** A function \( f \) from an open subset \( S \) of \( \mathbb{R} \) to \( \mathbb{R} \) is said to be continuous at a point \( x \in S \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for all \( y \in B_\delta(x) \).

**Exercise 1.37.** Let \( x_n \) be a converging sequence in \( S \). Show that if \( f \) is continuous at \( x \) then \( \lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) \).

**Definition 1.33.** A function is said to be continuous on \( S \) if it is continuous at each point of \( S \).

**Definition 1.34.** A function is said to be uniformly continuous on \( S \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for all \( x, y \in S \) with \( |x - y| < \delta \) it is true that \( |f(x) - f(y)| < \epsilon \).

Note that \( \delta \) is not allowed to depend on \( x \).

**Exercise 1.38.** Show that \( \sin(1/x) \) is not uniformly continuous on \((0, 1)\) even though it is continuous at every point on \((0, 1)\).

**Theorem 1.35.** If \( f: S \to \mathbb{R} \) is continuous on a compact set \( S \) then \( f \) is uniformly continuous on \( S \).

**Proof.** At each point \( x \in S \) pick a suitable a sufficiently small ball so that within that ball the function varies by no more than \( \epsilon \). These balls form an open cover. Pick a finite sub-cover and choose the smallest radius of the finite sub-cover as \( \delta \).

**Theorem 1.36.** If \( f \) is a continuous function on the compact set \( S \) then there is a point \( x \in S \) such that \( f(x) = \sup\{ f(y) | y \in S \} \).

**Proof.** First we need to show that \( f \) is bounded on \( S \). Suppose not. Then there is a sequence \( x_n \) in \( S \) with \( |f(x_n)| > \frac{1}{n} \). Pick a converging sub-sequence and show that \( f \) is not continuous at the limit point.

Let \( x_n \) be a sequence of points in \( S \) such that \( \lim f(x_n) = \sup_{S} f \). Then there is a converging sub-sequence of \( x_n \) in \( S \) and the continuity of \( f \) proves the theorem.

**Theorem 1.37.** Intermediate Value Theorem. Let \( f \) be continuous on \([a, b]\). Let \( c \) be a value between \( f(a) \) and \( f(b) \). Then there is an \( x \in [a, b] \) such that \( f(x) = c \).

**Proof.** Assume wolog that \( f(a) \leq c \leq f(b) \). Using bisection we can find a sequence of nested closed intervals that are shrinking to zero and the function is guaranteed to bracket \( c \) in each of these intervals. Now compactness and continuity yield the result. Exercise: fill in the details.

**Definition 1.38.** A sequence of continuous functions \( f_n: S \to \mathbb{R} \) is said to converge uniformly to the function \( f: S \to \mathbb{R} \), if for every \( \epsilon > 0 \) there is a \( N \) such that for all \( n > N \) and all \( x \in S \) \( |f(x) - f_n(x)| < \epsilon \).

**Theorem 1.39.** If the sequence of continuous functions \( f_n \) on the set \( S \) is converging uniformly to the function \( f \), then \( f \) is continuous on \( S \).

**Proof.** Pick \( n \) such that \( |f_n(x) - f(x)| < \epsilon \) for \( x \in S \). Pick \( \delta \) such that for \( |x - y| < \delta \), \( |f_n(x) - f(y)| < \epsilon \). Then

\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
\leq \epsilon + \epsilon + \epsilon.
\]

Read up on Riemann integration in the appendix of Stein & Shakarchi’s book. We will use it freely in the sequel.
Proposition 1.40. If \( f \) and \( g \) are integrable functions on \([a, b]\) and \( c \) is a real number then
\[
\int_a^b (f + cg) = \int_a^b f + c \int_a^b g.
\]

Theorem 1.41. If \( f \) is a continuous function on \([a, b]\) then it is integrable on \([a, b]\).

Proposition 1.42. If \( f \) is integrable on \([a, b]\) and \( |f(x)| \leq M \) on \([a, b]\) then \( \left| \int_a^b f \right| \leq M(b - a) \).

Proposition 1.43. If \( a < b < c \) then
\[
\int_a^c f = \int_a^b f + \int_b^c f
\]
holds whenever one side holds.

Theorem 1.44. Suppose \( f_n \) is a sequence of continuous functions that is converging uniformly to \( f \) on \([a, b]\), then
\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b \lim_{n \to \infty} f_n = \int_a^b f.
\]

Proof. \( f \) is continuous and hence integrable. Then
\[
\left| \int_a^b f_n - \int_a^b f \right| = \int_a^b |f_n - f| \leq \sup_{a \leq x \leq b} |f_n(x) - f(x)||a - b|,
\]
which proves the result. \( \square \)

Definition 1.45. A function \( f \) on \((a, b)\) is said to be differentiable at \( x \in (a, b) \) if
\[
\lim_{y \to x} \frac{f(x) - f(y)}{x - y}
\]
exists. The limit is denoted by \( f'(x) \) and is called the derivative of \( f \) at \( x \).

Proposition 1.46. If \( f \) is differentiable at \( x \) then it is continuous at \( x \).

Theorem 1.47. Mean Value Theorem. Let \( f : [a, b] \to \mathbb{R} \) be continuous and differentiable on \((a, b)\). Then for some \( c \in (a, b) \)
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Proof. Consider the function \( g(x) = f(x) - (f(a) \frac{x - b}{a - b} + f(b) \frac{x - a}{b - a}) \). Then \( g(a) = g(b) = 0 \) and \( g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \). We just need to show that there is a \( c \) for which \( g'(c) = 0 \). The candidate points are the points of maxima and minima which are guaranteed to exist. Let \( c^* \) be a point where \( g \) becomes maximum and \( c_* \) be a point where \( g \) becomes minimum. If both \( c^* \) and \( c_* \) are end-points then \( g = 0 \) and the proof is done. Wolog let \( c^* \in (a, b) \). Then \( g(x) \leq g(c^*) \). It follows that
\[
\frac{g(c^*) - g(x)}{c^* - x}
\]
is non-negative if \( x < c^* \) and non-positive if \( x > c^* \). It follows that the limit must be \( 0 = g'(c^*) \). \( \square \)