Fourier Analysis for Engineers

S. Chandrasekaran

March 31, 2009

Chapter 1
Basic real analysis

A good reference for this chapter is the book *Advanced Mathematical Analysis* by Richard Beals.

We will denote the set of real numbers by $\mathbb{R}$. We will assume that the basic properties of the real numbers are obvious. The one non-obvious property we will assume as an axiom.

Definition 1.1. *A number M is said to be an* **upper bound** *for a subset S of $\mathbb{R}$ if $s \leqslant M$ for all $s \in S$.*

Axiom 1.2. *If S is a subset of $\mathbb{R}$ with a finite upper bound, then there exists a number $\operatorname{lub}(S)$, called the* **least upper bound** *of S, which is an upper bound of S that is less than every other upper bound of S.*

Exercise 1.1. Show that for every $\epsilon > 0$ there is an $x \in S$ such that $x > \operatorname{lub}(S) - \epsilon$.

Exercise 1.2. Give an example of a bounded subset of the rational numbers that has no greatest lower bound.

Definition 1.3. *A number L is said to be a* **lower bound** *for a subset S of $\mathbb{R}$ if $s \geqslant L$ for all $s \in S$.*

Lemma 1.4. *If S is a subset of $\mathbb{R}$ with a finite lower bound, then there exists a number $\operatorname{glb}(S)$, called the* **greatest lower bound** *of S, which is a lower bound of S that is greater than every other lower bound of S.*

Proof. Exercise. *Hint*: Apply axiom 1.2 to $-S$. □

Notation 1.5. *The least upper bound of S is also called the* **supremum** *of S and denoted by $\sup(S)$. Similarly the greatest lower bound of S is also called the* **infimum** *of S and denoted by $\inf(S)$.*

Exercise 1.3. Show that $\inf(S) \leqslant \sup(S)$.

Definition 1.6. *Let $a < b$. Then the subset $\{x \,|\, a < x < b\}$ of $\mathbb{R}$ is called an* **open interval** *and denoted by (a, b).*

Definition 1.7. *An* **open ball** *centered at a of radius r is the open interval $(a - r, a + r)$ and is denoted by $B_r(a)$.*

Definition 1.8. *A* **neighborhood** *of a point is an open ball centered at that point.*

Definition 1.9. *A subset S of $\mathbb{R}$ is said to be* **open** *if every point in S has a neighborhood that is contained in S. The empty set is assumed to be open.*

Exercise 1.4. Show that an open interval is an open set.

Proposition 1.10. *The (possibly infinite) union of open sets is open.*

Proof. Exercise. □

Proposition 1.11. *The finite intersection of open sets is open.*

Proof. Exercise. □

Definition 1.12. *A point x is said to be a* **limit point** *of a subset S if every neighborhood of x also contains a point of S other than x itself.*

Exercise 1.5. Show that a and b are limit points of the open interval $(a.b)$.

Definition 1.13. *Let $a \leqslant b$. Then the subset $\{x \,|\, a \leqslant x \leqslant b\}$ of $\mathbb{R}$ is called a* **closed interval** *and denoted by $[a, b]$.*

Definition 1.14. *A subset of $\mathbb{R}$ is said to be* **closed** *if it contains all its limit points. The empty set is assumed to be closed.*

Exercise 1.6. Let $-\infty < a < b < \infty$. Show that $[a, b]$ is a closed set.

Exercise 1.7. Show that a finite subset of $\mathbb{R}$ has no limit points. Thus they are closed (vacuously).

Exercise 1.8. Show that if $x \notin S$ is not a limit point of S then there is a neighborhood of x that does not intersect S.

Exercise 1.9. Show that the finite union of closed sets is a closed set. *Hint*: Show that if x is not in the union then it has a ball, one for each closed set, that does not intersect that closed set. The intersection of these balls gives a non-intersecting ball. Thus the union is not missing a limit point.

Exercise 1.10. Show that the (possibly infinite) intersection of closed sets is a closed set.

Notation 1.15. *If S is a subset let $S^c = \{x \,|\, x \notin S\}$ denote the complement of S.*

Exercise 1.11. Show that $\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$, where S_n is a set. The number of sets can be infinite.

Exercise 1.12. Show that an open set can be written as the (possibly infinite) union of open intervals.

Proposition 1.16. *A subset is open iff its complement is closed.*

Proof. Exercise. □

Definition 1.17. *A sequence of numbers x_n for $n = 1, 2, \ldots,$ is said to have a* **limit** *x, if for every $\epsilon > 0$ there exists an N such that for all $n > N$, $|x_n - x| < \epsilon$. We denote this by $\lim_{n\uparrow\infty} x_n = x$, and say that the sequence x_n* **converges** *to x. If such an x does not exist we say that the sequence x_n* **diverges**.

Exercise 1.13. Show that the sequence $x_n = n$ for $n = 1, 2, \ldots,$ diverges.

Exercise 1.14. Show that the sequence $x_n = (-1)^n$ for $n = 1, 2, \ldots,$ diverges.

Exercise 1.15. Show that the limit of a sequence is unique if it exists.

Exercise 1.16. Show that a convergent sequence is bounded.

Exercise 1.17. Let x_n be a bounded sequence of non-decreasing real numbers; that is, there is an $M < \infty$ such that $|x_n| < M$, and $x_n \leqslant x_{n+1}$. Show that $\lim_{n\uparrow\infty} x_n$ exists. *Hint*: Look at the supremum of the set $\{x_n\}$.

Exercise 1.18. Show that if $a > 0$ then $\lim_{n\uparrow\infty} a^{1/n} = 1$. *Hint*: If $a < 1$ then this is an increasing sequence.

Exercise 1.19. Show that $\lim_{n\uparrow\infty} n^{1/n} = 1$. *Hint*: Let $n^{1/n} = 1 + x_n$. Then by the binomial expansion $n \geqslant 1 + n\,x_n + \frac{1}{2}n(n-1)x_n^2 \geqslant \frac{1}{2}n(n-1)x_n^2$.

Let x_n be a bounded real sequence and define $A_N = \{x_n \,|\, n \geqslant N\}$, $l_N = \inf(A_N)$ and $u_N = \sup(A_N)$.

Exercise 1.20. Show that l_N is a bounded non-decreasing sequence and u_N is a bounded non-increasing sequence. Therefore both $\lim_{N\uparrow\infty} l_N$ and $\lim_{N\uparrow\infty} u_N$ exist.

Exercise 1.21. Let $x_1 = -1$, $x_2 = 4$ and $x_n = n \bmod 4$ for $n = 3, 4, \ldots$. Find $\lim_{N\uparrow\infty} l_N$ and $\lim_{N\uparrow\infty} u_N$ for this sequence.

Definition 1.18. *Let x_n be a bounded real sequence. We call $\lim_{N\uparrow\infty} l_N$ as the* **lower limit** *of the sequence x_n and denote it by $\lim\inf x_n$. We call $\lim_{N\uparrow\infty} u_N$ as the* **upper limit** *of the sequence x_n and denote it by $\limsup x_n$.*

Exercise 1.22. Show that $l_N \leqslant u_M$.

Exercise 1.23. Show that $\liminf x_n \leqslant \limsup x_n$.

Exercise 1.24. Show that if $\liminf x_n < \limsup x_n$, then the sequence x_n diverges.

Exercise 1.25. Show that $\lim_{n\uparrow\infty} x_n = \liminf x_n = \limsup x_n$ iff the sequence is converging.

Definition 1.19. *A sequence of numbers x_n for $n = 1, 2, \ldots,$ is said to be a* **Cauchy sequence** *if for every $\epsilon > 0$, there is an integer N such that for all $n, m > N$, $|x_n - x_m| < \epsilon$.*

Exercise 1.26. Show that if a sequence of real numbers converges then it is a Cauchy sequence.

Exercise 1.27. Show that a Cauchy sequence is bounded.

Exercise 1.28. Show that for a Cauchy sequence $\liminf x_n = \limsup x_n$. Therefore a Cauchy sequence of real numbers must converge.

Let x_n be a real sequence. Let $S_N = \sum_{n=1}^{N} x_n$ be the N-th partial sum of the infinite series $\sum_{n=1}^{\infty} x_n$. We say that the infinite series converges to s if $\lim_{N\uparrow\infty} S_N = s$. We denote this by $\sum_{n=1}^{\infty} x_n = s$. Othewise we say that the infinite series diverges.

While this definition seems obvious and well-defined, it is not the case. The following two examples are from Beals.

Example 1.20. The series $\sum_{n=1}^{\infty} n^{-1}$ diverges. To see this observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots \\ &\geqslant \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots \\ &= \frac{1}{2} + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} + \ldots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \\ &= \infty.\end{aligned}$$

Exercise 1.29. Show that if $|x| < 1$ then $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$.

Exercise 1.30. Riemann. Show that the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ can be rearranged to converge to any real number.

This is the reason why convergence as we have defined it for series can be problematic.

Example 1.21. The series $\sum_{n=1}^{\infty} n^{-2}$ converges. To see this observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{1}{8}\right)^2 + \ldots \\ &\leqslant 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \ldots \\ &= 1 + 2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{4}\right)^2 + 8\left(\frac{1}{8}\right)^2 + \ldots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \\ &= 2.\end{aligned}$$

Exercise 1.31. Show that if the sum $\sum_n x_n$ converges then $\lim_{n\uparrow\infty} x_n = 0$. *Hint*: The partial sums are a Cauchy sequence. Consider $|S_n - S_{n+1}|$.

Exercise 1.32. Generalize the above two examples to show that if $0 \leqslant a_{n+1} \leqslant a_n$, then $\sum a_n$ converges iff $\sum 2^k a_{2^k}$ converges.

Exercise 1.33. Show that $\sum n^{-1}(\log n)^{-2}$ converges while $\sum n^{-1}(\log n)^{-1}$ diverges.

Exercise 1.34. Let $0 \leqslant a_{n+1} \leqslant a_n$ and let f be a monotonic non-increasing function such that $f(n) = a_n$. Show that $\sum a_n$ converges iff $\int_1^{\infty} f(x)dx$ converges.

Exercise 1.35. Show that $\sum a_n$ converges for $a_n = n^{-p}$ if $p > 1$ and diverges if $p \leqslant 1$.

Proposition 1.22. Comparison test. *Let $\sum_n a_n$ be a positive converging series. Then if $|x_n| \leqslant a_n$, then $\sum_n x_n$ is also converging.*

Proof. Exercise. □

Proposition 1.23. Ratio test. *(a) If* $\limsup |z_{n+1}/z_n| < 1$ *then* $\sum z_n$ *converges. (b) If* $\lim \inf |z_{n+1}/z_n| > 1$ *then* $\sum z_n$ *diverges.*

Proof. (a). For sufficently large N $|z_{n+1}/z_n| \leqslant r < 1$. Exercise. Finish the proof. (b). Observe that z_n cannot approach 0. □

Exercise 1.36. Show that $\sum z^n/n!$ converges for all values of z.

Definition 1.24. *Let S be a subset of $\mathbb{R}$. A family A_n of sets is said to* **cover** *S if $S \subset \bigcup_n A_n$.*

Definition 1.25. *A subset S of $\mathbb{R}$ is said to be* **compact** *if every open cover of S has a finite sub-cover. That is, if the family of open sets A_n covers S, then there are sets A_{n_1}, A_{n_2}, ..., A_{n_m} such that $S \subset \bigcup_{k=1}^m A_{n_k}$.*

Example 1.26. $\mathbb{R}$ is not compact since the family $A_n = (n, n+2)$ has no finite sub-cover.

Example 1.27. The set (0, 1) is not compact. The open cover $A_n = \left(\frac{1}{n}, 1\right)$ has no finite sub-cover.

Example 1.28. A finite subset of $\mathbb{R}$ is compact.

Theorem 1.29. Bolzano-Weierstrass. *Let x_n be a sequence in a closed and bounded subset S of $\mathbb{R}$. Then there is a sub-sequence of x_n that converges to a point in S.*

Proof. If there are only finite number of distinct points in x_n there is nothing to prove. So assume otherwise. Since S is bounded we can find a closed interval $I_0 = [a_0, b_0]$ that covers S. Since there are infinitely many distinct points in x_n one of the two intervals $[a_0, \frac{a_0 + b_0}{2}]$ and $[\frac{a_0 + b_0}{2}, b_0]$ must contain infinitely many points of x_n. Call that interval $I_1 = [a_1, b_1]$. Pick a point of the sequence x_{n_1} in I_1 and remove it from further consideration. Proceed thus to construct $I_k = [a_k, b_k]$ and a correspoding point of the sequence x_{n_k} in I_k. Note that $\lim b_k - a_k = 0$, since the intervals are halved every time and we started with an interval of finite length. Clearly the sequence we constructed is Cauchy. Exercise: finish the proof. □

Theorem 1.30. Heine-Borel theorem. *A subset of $\mathbb{R}$ is compact iff it is closed and bounded.*

Proof. Suppose S is a compact subset. For each $x \in S$ let $A_x = B_1(x)$. Since this is an open cover of S there is a finite sub-cover from it. From this it follows that S is bounded. Exercise: provide the details.

Let $x \in S^c$. For every $y \in S$ there is a ball centered at y that does not include x. These provide an open cover of S. There is a finite sub-cover from these balls none of which intersect x. Hence there is a ball at x that does not intersect the finite sub-cover, and hence does not intersect S. Therefore S^c is open.

To prove the converse let S be a closed and bounded subset of $\mathbb{R}$. Let A_n be an open cover of S. For each $x \in A_n$ take every ball centered at x that is entirely contained in A_n. The set of all such balls forms an open cover of S. Clearly if we can prove that there is a finite sub-cover from this set we can get a corresponding finite sub-cover for A_n. So we restrict our attention to these open balls at each $x \in S$. For each $x \in S$ consider the supremum of the radii of all balls centered at x that are in the open cover and pick a ball whose radius is at least half this supremum. Call this ball at x as $B_{r(x)}(x)$. We claim that the infimum of $r(x)$ is strictly greater than 0. Suppose it is not. Then the infimum is 0. This implies that there is a sequence in x_n with $\lim_{n\uparrow\infty} r(x_n) = 0$. Pick a converging sub-sequence of x_n and call that as x_n. Let $x = \lim_{n\uparrow\infty} x_n \in S$. But there is a finite open ball at x in the open cover. Which implies that for every x_n in this ball there is a ball of sufficiently large radius. Therefore the infimum cannot be 0 since we tried to pick large enough balls. Therefore for every $x \in S$ we can pick a ball in the open cover whose radius is bigger than some $\delta > 0$. Exercise: from this show that we can pick a finite number of open balls to cover S since it is bounded. □

Example 1.31. The set $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is compact since it is closed and bounded.

Definition 1.32. *A function f from an open subset S of $\mathbb{R}$ to $\mathbb{R}$ is said to be* **continuous at a point** $x \in S$ *if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in B_\delta(x)$.*

Exercise 1.37. Let x_n be a converging sequence in S. Show that if f is continuous at x then $\lim_{n\uparrow\infty} f(x_n) = f(\lim_{n\uparrow\infty} x_n)$.

Definition 1.33. *A function is said to be* **continuous** *on S if it is continuous at each point of S.*

Definition 1.34. *A function is said to be* **uniformly continuous** *on S if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$ it is true that $|f(x) - f(y)| < \epsilon$.*

Note that δ is not allowed to depend on x.

Exercise 1.38. Show that $\sin(1/x)$ is not uniformly continuous on $(0, 1)$ even though it is continuous at every point on $(0, 1)$.

Theorem 1.35. *If $f: S \to \mathbb{R}$ is continuous on a compact set S then f is uniformly continuous on S.*

Proof. At each point $x \in S$ pick a suitable a sufficiently small ball so that within that ball the function varies by no more than ϵ. These balls form an open cover. Pick a finite sub-cover and choose the smallest radius of the finite sub-cover as δ. □

Theorem 1.36. *If f is a continous function on the compact set S then there is a point $x \in S$ such that $f(x) = \sup\{f(y) | y \in S\}$.*

Proof. First we need to show that f is bounded on S. Suppose not. Then there is a sequence x_n in S with $|f(x_n)| > \frac{1}{n}$. Pick a converging sub-sequence and show that f is not continuous at the limit point.

Let x_n be a sequence of points in S such that $\lim f(x_n) = \sup_S f$. Then there is a converging sub-sequence of x_n in S and the continuity of f proves the theorem. □

Theorem 1.37. Intermediate Value Theorem. *Let f be continuous on $[a, b]$. Let c be a value between $f(a)$ and $f(b)$. Then there is an $x \in [a, b]$ such that $f(x) = c$.*

Proof. Assume wolog that $f(a) \leqslant c \leqslant f(b)$. Using bisection we can find a sequence of nested closed intervals that are shrinking to zero and the function is guaranteed to bracket c in each of these intervals. Now compactness and continuity yield the result. Exercise: fill in the details. □

Definition 1.38. *A sequence of continuous functions $f_n: S \to \mathbb{R}$ is said to converge uniformly to the function $f: S \to \mathbb{R}$, if for every $\epsilon > 0$ there is a N such that for all $n > N$ and all $x \in S$ $|f(x) - f_n(x)| < \epsilon$.*

Theorem 1.39. *If the sequence of continuous functions f_n on the set S is converging uniformly to the function f, then f is continuous on S.*

Proof. Pick n such that $|f_n(x) - f(x)| < \epsilon$ for $x \in S$. Pick δ such that for $|x - y| < \delta$, $|f_n(x) - f(y)| < \epsilon$. Then

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leqslant \epsilon + \epsilon + \epsilon. \end{aligned}$$

□

Read up on Riemann integration in the appendix of Stein & Shakarchi's book. We will use it freely in the sequel.

Proposition 1.40. *If f and g are integrable functions on $[a,b]$ and c is a real number then*

$$\int_a^b (f+cg) = \int_a^b f + c\int_a^b g.$$

Theorem 1.41. *If f is a continuous function on $[a,b]$ then it is integrable on $[a,b]$.*

Proposition 1.42. *If f is integrable on $[a,b]$ and $|f(x)| \leqslant M$ on $[a,b]$ then $\left|\int_a^b f\right| \leqslant M(b-a)$.*

Proposition 1.43. *If $a<b<c$ then*

$$\int_a^c f = \int_a^b f + \int_b^c f$$

holds whenever one side holds.

Theorem 1.44. *Suppose f_n is a sequence of continuous functions that is converging uniformly to f on $[a,b]$, then*

$$\lim_{n\uparrow\infty}\int_a^b f_n = \int_a^b \lim_{n\uparrow\infty} f_n = \int_a^b f.$$

Proof. f is continuous and hence integrable. Then

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b f_n - f\right| \leqslant \sup_{a\leqslant x\leqslant b}|f_n(x)-f(x)|\,|a-b|,$$

which proves the result. □

Definition 1.45. *A function f on (a,b) is said to be* **differentiable** *at $x\in(a,b)$ if*

$$\lim_{y\to x}\frac{f(x)-f(y)}{x-y}$$

exists. The limit is denoted by $f'(x)$ and is called the **derivative** *of f at x.*

Proposition 1.46. *If f is differentiable at x then it is continuous at x.*

Theorem 1.47. Mean Value Theorem. *Let $f\colon[a,b]\to\mathbb{R}$ be continuous and differentiable on (a,b). Then for some $c\in(a,b)$*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Consider the function $g(x) = f(x) - \left(f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}\right)$. Then $g(a) = g(b) = 0$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. We just need to show that there is a c for which $g'(c)=0$. The candidate points are the points of maxima and minima which are guaranteed to exist. Let c^* be a point where g becomes maximum and c_* be a point where g becomes minimum. If both c^* and c_* are end-points then $g=0$ and the proof is done. Wolog let $c^*\in(a,b)$. Then $g(x)\leqslant g(c^*)$. It follows that

$$\frac{g(c^*)-g(x)}{c^*-x}$$

is non-negative if $x<c^*$ and non-positive if $x>c^*$. It follows that the limit must be $0 = g'(c^*)$. □