Partial Feedback Linearization (PFL) (Spong'94)

For an UNDERACTUATED system, where

\[ m < n \]

\[ \frac{\text{# actuated DoF}}{\text{# DoF total}} \]

- There remain \[ k = n - m \] "passive joints".

- If we have a GOOD MODEL of the dynamics,
  - and if \( m \geq k \) (more actuated than passive joints)
  - and if we can guarantee we can calculate any required
    matrix inverse or pseudo-inverse ("STRONG INERTIAL COUPLING")

- Then
  - We can "shuffle around" the terms in the equations
    of motion to solve DIRECTLY for ANY \( m \) of \( n \)
    DoF's — not just the so-called "active" joints.

General Case:

Nonlinear equations of motion (EOM) may be written as:

1. \[ \begin{bmatrix} M_{ii} \dot{q}_i + M_{ij} \dot{q}_j + h_i + \phi_i = 0 \end{bmatrix} \]

2. \[ \begin{bmatrix} M_{ij} \dot{q}_j + M_{jj} \dot{q}_j + h_j + \phi_j = 0 \end{bmatrix} \]

To control some set \( m \) of the \( n \) DoF, of joints:

- "Linearize" these non-linear EOM at each "dt" in time (as controller runs),
  - Plug in current \( q_i \) and \( \dot{q}_i \) values, for all \( i \).
- Come up with a control law (e.g., simple "PD" proportional
  plus derivative control) that defines \( m \) of \( n \) desired accelerations:
  \[ \ddot{q}_i \rightarrow (\text{for } m \text{ DoF's}) \rightarrow \text{pick } \ddot{q}_i \text{ to get toward desired } q_i(\ddot{q}) \ldots \]
- Use matrix algebra to solve for \( m \) elements in \( \tau \) ("torques")
  to achieve \( m \) values, \[ \ddot{q}_i = \ddot{q}_i^{\text{des}} \rightarrow \text{"Shuffle" EOM, to solve for } \tau. \]
ACROBOT Example

Define some terms:

\[ m_1 = m_1 l_1^2 + m_1 l_2^2 + 2m_1 l_1 l_2 \cos(q_2) + I_1 + I_2 \]

\[ m_2 = m_2 l_2^2 + I_2 \]

\[ m_{12} = m_2 l_1 l_2 \cos(q_2) + I_2 \]

\[ h_1 = m_2 l_1 l_2 \sin(q_2) \]

\[ h_2 = m_2 l_2 \sin(q_1) \]

\[ \phi_1 = (m_1 + m_2 l_1) g \cos(q_1) + m_2 l_2 g \cos(q_1 + q_2) \]

\[ \phi_2 = m_2 l_2 g \cos(q_1 + q_2) \]

\[ \| \cdot \| \]

Collocated \( \tau_2 \) set by choosing \( \dot{q}_2 \)

Non-collocated \( \tau_2 \) set by choosing \( \dot{q}_1 \)

1. \( m_1 \ddot{q}_1 + m_{12} \ddot{q}_2 + h_1 + \phi_1 = 0 \)

2. \( m_{21} \ddot{q}_1 + m_{22} \ddot{q}_2 + h_2 + \phi_2 = \tau_2 \)

A. Rewrite 1. w/ \( \dot{q}_1 \) on LHS:

\[ \dot{q}_1 = \frac{-1}{m_{11}} (m_{12} \ddot{q}_2 + h_1 + \phi_1) \]

B. Plug this in for \( \ddot{q}_1 \) in 2.:

\[ m_{21} \left[ \frac{-1}{m_{11}} (m_{12} \ddot{q}_2 + h_1 + \phi_1) \right] + m_{22} \ddot{q}_2 + h_2 + \phi_2 = \tau_2 \]

C. Drive \( \ddot{q}_2 \) toward some desired \( \ddot{q} \):

\[ \ddot{q}_2 = V \frac{\ddot{q}_2}{T} \]

D. Control law (PD) sets \( \ddot{q} \) to move \( \dot{q}_2 \) toward \( \ddot{q} \):

\[ \ddot{q}_2 = V \left[ K_p (\ddot{q} - \ddot{q}_2) - K_d \dot{q}_2 \right] \]

E. Plug \( \ddot{q}_2 \) from D. into B. to get \( \tau_2 \)!!
General Case \((m > 1, \ell > 1)\) \(\text{MATRIX equations!}\)

Collocated
- Both start with the same cone, in matrix form

Non-collocated
- Collocated

1. \(M_{12} \ddot{q}_i + M_{12} \ddot{q}_j + h_i + \phi_i = 0\)
2. \(M_{2i} \ddot{q}_i + M_{22} \ddot{q}_j + h_j + \phi_j = T\)

Plug in:
\[
\ddot{q}_i = -M_{12}^{-1}(M_{12} \ddot{q}_j + h_i + \phi_i)
\]

Plug in:
\[
\ddot{q}_j = -M_{12}^+ (M_{12} \ddot{q}_i + h_i + \phi_i)
\]

Where \(M_{12}^+\) is the PSEUDO- INVERSE of the \((\ell \times m)\) matrix \(M_{12}\):
\[
M_{12}^+ = M_{12}^T (M_{12} M_{12}^T)^{-1}
\]

A. Then 2. becomes:
\[
M_{2i} \ddot{q}_i + M_{22} \ddot{q}_j + h_i + \phi_i = T
\]

A. Then 2. becomes:
\[
M_{2i} \ddot{q}_i - M_{22} \ddot{q}_j (M_{12} \ddot{q}_i + h_i + \phi_i) + h_j + \phi_j = T
\]

B. Collecting terms: (collocated case) \((\text{Eq. 11})\)
\[
(M_{22} - M_{2i} M_{12}^T M_{12} M_{22}^T) \ddot{q}_j + (h_2 - M_{2i} M_{12}^T h_i) + (\phi_2 - M_{2i} M_{12}^T \phi_i) = T
\]

\[
\ddot{q}_j = V_2
\]

(\text{Eq. 33})

\[
\ddot{q}_i = V_1
\]

B. Collecting terms: (non-collocated case) \((\text{Eq. 35})\)
\[
(M_{21} - M_{22} M_{12}^T M_{21}^T) \ddot{q}_j + (h_2 - M_{22} M_{12}^T h_i) + (\phi_2 - M_{22} M_{12}^T \phi_i) = T
\]

(\text{Eq. 33})

C. Write a control law to set \(V_2\).

C. Write a control law to set \(V_1\)