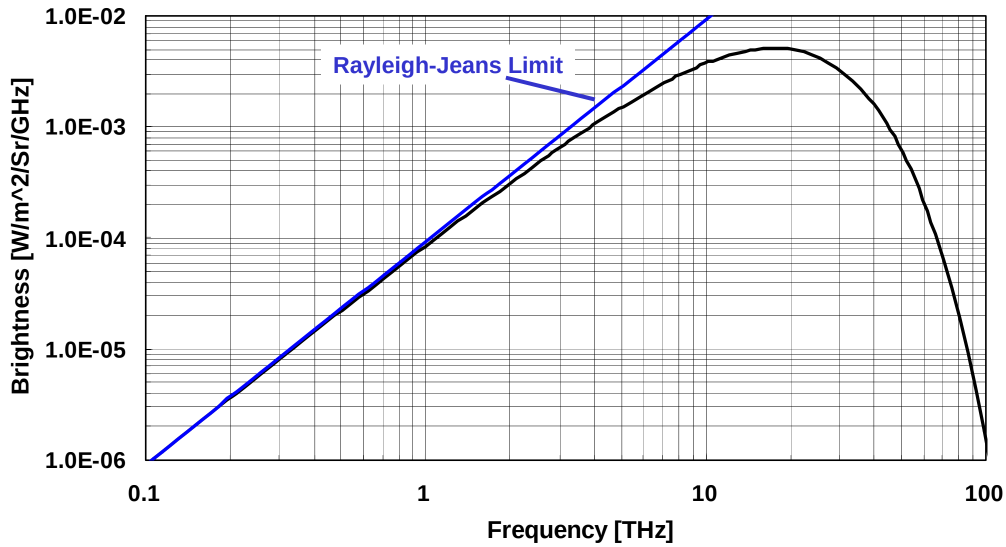
ECE594I, Prof. Brown, Fall Quarter 2009

Solutions #1




Brightness function for Problem 1 (unity emissivity and T = 300 K).

1.(a) From the course lecture notes, the brightness function is given by $\frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/k_BT}-1}$ [W/m$^2$/Sr]. A plot of this brightness function vs frequency is shown below for T = 300 K. (a) by inspection and iteration, it is easy to find that the maximum value of B occurs at 17.6 THz ($\lambda$ = 17.0 $\mu$m). (b) The maximum brightness function is 5.105x10$^{-12}$ W/(m$^2$-Hz). (c) The brightness function at $\nu$ = 1.0 THz is 8.48x10$^{-14}$ W/(m$^2$-Hz), 60.2 times lower than the maximum !

2. The integrated brightness function is defined and evaluated as follows:

$$IB = \int_0^\infty \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/k_BT}-1} = \frac{2h}{c^2}\left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{x^3dx}{e^x-1} = \frac{2h}{c^2}\left(\frac{k_BT}{h}\right)^4 \frac{\pi^4}{15} \equiv \sigma T^4/\pi$$

where $\sigma$ = 5.67x10$^{-8}$ W/(m$^2$-K$^4$) [note the good mnemonic ordering of numbers] (b) At T = 300 K, IB = 146.2 W/(m$^2$-Sr). (c) In the (long-wavelength) Rayleigh-Jeans limit, $B_T = 2k_BT(\nu/c)^2$, so that

$IB(<\nu_C) = \int_0^{\nu_C} \frac{2k_BT\nu^2}{c^2} d\nu = 2k_BT(\nu_C)^3/3c^2$. And using elementary calculus, we find

$IB(>\nu_C) = \int_{\nu_C}^\infty B(\nu)d\nu = \int_0^\infty B(\nu)d\nu - \int_0^{\nu_C} B(\nu)d\nu = \sigma T^4/\pi - 2k_BT(\nu_C)^3/3c^2$. (d) For $\nu_C$ = 100 GHz, IB ($<\nu_C$) = 0.031 mW/(m$^2$-Sr), which is a tiny fraction 2.1x10$^{-7}$ of the total IB.

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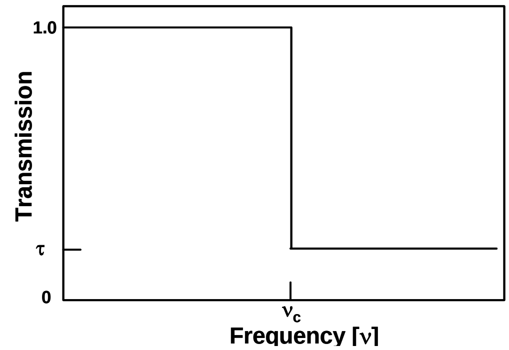
3. (a) First, it is helpful to visualize the given transmission function $T = |S_{21}|^2 = \tau\cdot\theta(\nu - \nu_C) + \theta(\nu_C - \nu)$, which is plotted to the right. Because $\nu_C$ satisfies the Rayleigh-Jeans criterion, the integrated brightness up to $\nu_C$ is




$IB(<\nu_C) = \int_0^{\nu_C} \frac{2k_B T\nu^2}{c^2} d\nu = 2k_B T(\nu_C)^3/3c^2$. From Problem 1, the total integrated brightness is $\sigma T^4/\pi$. So the filter transmission function imposes the condition

$\tau[\sigma T^4/\pi - 2k_B T(\nu_c)^3/3c^2] = 2k_B T(\nu_c)^3/3c^2$ or

$\tau = 2k_B T(\nu_c)^3/3c^2\ [\sigma T^4/\pi - 2k_B T(\nu_c)^3/3c^2]^{-1}$

(b) evaluation of this expression for T = 300 and $\nu_C$ = 1.0 THz yields $\tau = 2.1\text{x}10^{-4}$, which is a challenging filter to design. In decibel units, we have $\tau$ [dB] = $10*\log_{10}(\tau)$ = -37 dB.

(c) From the perspective of Kirchoff's law, the scattering filter is better than the absorptive filter since the latter will absorb most of the radiation at $\nu > \nu_C$ and then re-emit it on the detector side. The scattering filter will disperse the incoming mid-IR radiation isotropically so that very little will get to the detector element.

4. (a) Variance for photons

$\langle(\Delta n_K)^2\rangle = \langle(n-\langle n_K\rangle)\rangle^2\rangle = \langle(n_K)^2\rangle - \langle n_K\rangle^2$, where $\langle n_K\rangle = [\exp(h\nu/k_BT)-1]^{-1}$ is the Planck function as derived from the Boltzman (exponential) density function

$$\langle n_K\rangle = \frac{\sum_{n_K=0}^{\infty} n_K \exp[-n_K h\nu/k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu/k_B T]} \quad \text{and} \quad \langle(n_K)^2\rangle = \frac{\sum_{n_K=0}^{\infty} n_K^2 \exp[-n_K h\nu/k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu/k_B T]}$$

This is best evaluated with the trick identity

$n_K^2 = \left(\frac{k_B T}{h}\right)^2 \left(\frac{d^2}{d\nu^2}\exp(-n_K h\nu/k_B T\right)\cdot\exp(nh\nu/k_B T)$, so the numerator above becomes

$$\langle(n_K)^2\rangle = (k_B T/h)^2 \sum_{n_K=0}^{\infty} (d^2/d\nu^2)\exp(-n_K h\nu/k_B T) = (k_B T/h)^2 (d^2/d\nu^2) \sum_{n_K=0}^{\infty} \exp(-n_K h\nu/k_B T)$$

$$= (k_B T/h)^2 (d^2/d\nu^2)\left[\frac{1}{1-\exp(-h\nu/k_B T)}\right] = \frac{\exp(-h\nu/k_B T)[1+\exp(-h\nu/k_B T)]}{[1-\exp(-h\nu/k_B T)]^3}$$

Thus, we have $\langle(n_K)^2\rangle = \frac{\sum_{n_K=0}^{\infty} n_K^2 \exp[-n_K h\nu/k_B T]}{\sum_{n_K=0}^{\infty} \exp[-n_K h\nu/k_B T]} = \frac{\frac{\exp(-h\nu/k_B T)[1+\exp(-h\nu/k_B T)]}{[1-\exp(-h\nu/k_B T)]^3}}{\frac{1}{1-\exp(-h\nu/k_B T)}}$

$$<(n_K)^2> = \frac{\exp(-h\nu/k_BT)[1+\exp(-h\nu/k_BT)]}{[1-\exp(-h\nu/k_BT)]^2} = \frac{\exp(h\nu/k_BT)[1+\exp(-h\nu/k_BT)]}{[\exp(h\nu/k_BT)-1]^2} = \frac{\exp(h\nu/k_BT)+1)}{[\exp(h\nu/k_BT)-1]^2}$$

$$<(n_K)^2> - <n_K>^2 = \frac{\exp(h\nu/k_BT)+1}{[\exp(h\nu/k_BT)-1]^2} - \frac{1}{[\exp(h\nu/k_BT)-1]^2} = \frac{\exp(h\nu/k_BT)}{[\exp(h\nu/k_BT)-1]^2}$$

$= <n_K>^2 (<n_K>^{-1}+1) = <n_K>(<n_K>+1)$ {The famous “photon bunching” expression}

(b) For the sun radiation at $T_B$ = 5800 K in a single spatial mode:
• at $\lambda$ = 0.5 mm, $\nu$ = 0.6 THz, $<n_K>$ = 200.8, $<(\Delta n_K)^2>$ = 4.05x10$^4$
$<P_K> = <n_K>h\nu\Delta\nu$ = 8.0x10$^{-11}$ W ; $<(\Delta P_K)^2> = <(\Delta n_K)^2>(h\nu\Delta\nu)^2$ = 6.4x10$^{-21}$ W
so $(P_K)_{rms}$ = 8.0x10$^{-11}$ W = -71 dBm

• at $\lambda$ = 0.5 micron, $\nu$ = 6x10$^{14}$ Hz, $<n_K>$ = 7.0x10$^{-3}$, $<(\Delta n_K)^2> \approx$ 7.0x10$^{-3}$
$<P_K> = <n_K>h\nu\Delta\nu$ = 2.8x10$^{-12}$ W; $<(\Delta P_K)^2> = <(\Delta n_K)^2>(h\nu\Delta\nu)^2$ = 1.1x10$^{-21}$ W
so $(P_K)_{rms}$ = 3.3x10$^{-11}$ W = -75 dBm.