HW#3 Solutions

1) To evaluate the feedhorn, start with $F(\theta,\phi) = \exp(-10\,\theta^2)$,

(a) Half-power beam width defined by $0.5 = \exp[-10(\beta/2)^2]$ where β is the full width at the half-power points.

Solving f or β, we get $2*[-\ln(0.5)/10]^{1/2}$ = 0.53 rad = 30.2°.

(b) The pattern solid angle is $\Omega_p = \iint F(\theta,\phi)\sin\theta d\theta d\phi$

But this is not integrable in θ (try mathematically to verify this using a symbolic integration code, such as Mathmatica or Maple). So we take advantage of fact that the pattern given has but a single main lobe, and approximate by spherical trigonometry $\Omega_P \approx \beta_X \beta_Y$. From part (a) $\beta = \beta_X = \beta_Y = 0.527$, so $\Omega_P = 0.277$ steradians. But a better way to estimate Ω_P (as discussed in lecture) is possible since the beam is symmetric

$$\int_0^{2\rho}\int_0^{\pi} F(\theta,\phi)\sin\theta d\theta d\phi = \int_0^{2\pi}\cdot\int_0^{\beta/2} \sin\theta d\theta d\phi$$

where $F(\theta,\phi)$ is approximated as unity over the cone full angle β. This so called "ice-cream-cone" approximation leads to $\Omega_P \approx -2\pi\cos\theta\ \Big|_0^{\beta/2}$ = 2π(1-cosβ/2) = 0.217 steradian.

(c) The antenna directivity is:

$D = 4\pi/\Omega_P \approx 4\pi/0.277 = 45.4$ (according to pencil beam approximation)

$D = 4\pi/0.217 = 57.9$ (according to "ice-cream-cone" approximation)

(d) S_{max} at range of 10 cm is just $P_{rad}/(4\pi r^2)\cdot D = 4.6\text{x}10^{-5}$ W/cm^2 in "ice-cream-cone" approximation.

2) Dish with circular beamwidth of 1.5°.

(a) For pencil-beam case, $D \approx 4\pi/(\beta x \beta y)$. For the dish, $\beta = 1.5*\pi/180^o = 0.026$ rad.

So, $D \approx 4\pi/(0.026)^2 = 1.84\text{x}10^4$ or D = 10log10[$1.84\text{x}10^4$] = 42.6 dB

(b) If dish area is doubled, we expect the directivity to also double consistent with $D_{max} = 4\pi A/\lambda^2$, and the beamwidth should decrease consistent with $4\pi/(\beta_X\beta_Y) = 4\pi A/\lambda^2$. So new directivity $D = 3.66\times10^4 \Rightarrow 45.6dB$. And the new beamwidth $\beta = 1.5^o/(2)^{1/2} = 1.06^o = 0.018$ rad.

(c) If antenna frequency is doubled to 800 GHz, new directivity should go up four times consistent with $D_{max} = 4\pi A/\lambda^2 = 4\pi A f^2/c^2$. So the new directivity is $D = 7.32\text{x}10^4$ or 48.6 dB. And the new beamwidth should decrease by a factor of two consistent with $4\pi/(\beta^2) = 4\pi A f^2/c^2$, so $\beta = 1.5^o/2 = 0.75^o = 0.013$ rad.

(d) If used as the primary antenna in a receiver, the available power is just $P_{avail} = S_{Tx}*A_{eff} = S_{Tx}*D_{Rx}\lambda^2/(4\pi)$ where S_{Tx} is the transmitter radiated intensity at the receiver. From antenna theory applied to the transmit antenna, the maximum S_{Tx} (at the pattern maximum) is just $D_{Tx}*P_{rad}/(4\pi r^2)$ where P_{rad} is the total radiated power. Overall we have

$$P_{avail} = Prad*D_{Tx}*D_{Rx}\lambda^2/(4\pi r)^2 ,$$

(Friis' transmission formula). The best estimate of from 3(c) is D = 57.9, so with P_{rad} = 1 mW and r = 1 km, we find the maximum $S_{Tx} = 4.6\text{x}10^{-9}$ W/m^2. So with $D_{Rx} = 1.84\text{x}10^4$ from 2(a) and λ = 0.75 mm (400 GHz), we find $P_{avail} = 3.8\text{x}10^{-12}$ W.

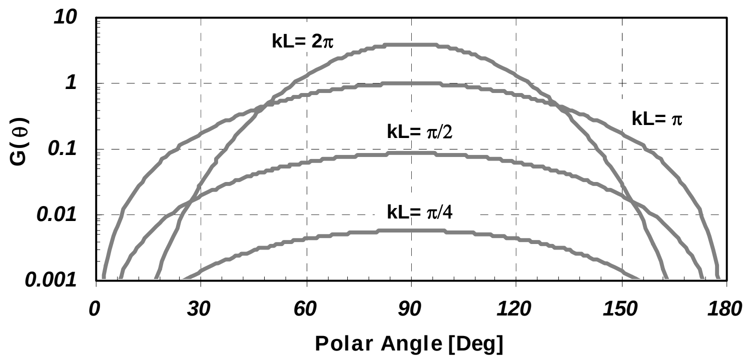

Fig. 1.

(e) We assume the dish is unimodal, so that according to the Rayleigh-Jeans limit of the Johnson-Nyquist theorem, $P_{noise} = k_B T\Delta\nu = 4.1\text{x}10^{-11}$ W for T = 300 K and $\Delta\nu$ = 10 GHz. Setting this noise power equal to the P_{avail} in Friis' formula and solving analytically for r, we find

$$r = \{Prad*D_{Tx}*D_{Rx}\lambda^2/[(4\pi)^2\ (k_B T\Delta\nu)]\}^{1/2}$$

which is called the "range" in radar and communications theory. When evaluated under the above conditions, we find r = 303 m.

3. (a). The key point is that even up to THz frequencies, the electrical conductivity of SI GaAs is so low that it can be treated as an ideal dielectric. From general electrodynamics, the intrinsic impedance is then defined by $\eta = \eta_0/(\varepsilon_r)^{1/2}$, where ε_r is the low-frequency. Looking in any good book on semiconductors, we find ε_r = 12.8, so η = 105.4 Ω, and the average dielectric constant at the interface is $\varepsilon_{ave} = (1 + \varepsilon_r)/2 = 6.9$.
(b). The interface velocity is just $v = c/(\varepsilon_{ave})^{1/2} = c/n_{ave} = 1.14\text{x}10^{10}$ cm/s. So the approximate physical length of a dipole antenna on the surface of the GaAs that will display a half-wave resonance at 600 GHz is $L_{half} = \lambda_{ave}/2 = v_{ave}/(2f)$ = 95 micron. By the same reasoning, the approximate physical length of a full-wave dipole antenna at 600 GHz is $L_{full} = \lambda_{ave} = v_{ave}/(f)$ = 190 micron.
(c) An inspection of this terms shows that it always displays a maximum at $\theta_d = \pi/2$ from L = 0 to at least L = λ, and the magnitude of the maximum increases monotonically from 0 (kL = 0) to 1 (kL = π; half wave) to 4 (kL = 2π; full wave). A graphical proof of these facts is shown in Fig. 1, as computed using Excel.
(d) By definition, the beam pattern function has to include all the angular dependence of the radiation and be normalized with at the angle in space where the time-averaged Poynting vector displays its maximum. According to the expression given for an arbitrary length dipole, all the angular dependence must be in the term

$$f(\theta) = \left[\frac{\cos[(kL\cos\theta_d)/2] - \cos kL/2}{\sin\theta_d}\right]^2$$

This becomes the beam pattern function $F(\theta,\phi)$once it is normalized to the maximum at $\theta_d = \pi/2$:

$$F(\theta) = \frac{1}{[1-\cos(kL/2)]^2}\left[\frac{\cos[(kL\cos\theta_d)/2] - \cos(kL/2)}{\sin\theta_d}\right]^2$$

4. (a) The total radiated power is found by integrating the Poynting vector from $\phi = 0$ to π, assuming that the radiation propagates into the substrate side only)

$$P_{rad} = \frac{I_0^2 \cdot \eta}{4\pi^2} \int_0^{\pi} \cdot \int_0^{\pi} \left[\frac{\cos[(kL\cos\theta_d)/2] - \cos kL/2}{\sin\theta_d} \right]^2 \sin\theta_d \, d\theta_d \cdot d\phi$$

$$= \frac{I_0^2 \cdot \eta}{4\pi} \int_0^{\pi} \left[\frac{\cos[(kL\cos\theta_d)/2] - \cos kL/2}{\sin\theta_d} \right]^2 \sin\theta \cdot d\theta$$

In general this integral can not be reduced to closed form, so a numerical method is required. The use of the Matlab function, quadl.m, for example results in the θ integral being 1.219 for L = λ/2 (kL = π) or equal to 3.318 for L = λ (kL = 2π). So the total radiated power is

$$P_{rad} = 10.2 \cdot I_0^2 \text{ for } L = \lambda/2$$
$$\text{and } P_{rad} = 27.8 \cdot I_0^2 \text{ for } L = \lambda.$$

(b) The directivity is found from D = $\dfrac{<\vec{S}>_{\max}}{P_{rad}/4\pi r_d^2} = \dfrac{(I_0^2\eta/4\pi^2 r_d^2)\cdot 1 \cdot 4\pi r_d^2}{10.2 \cdot I_0^2} = 3.29$ for L = λ/2 and D = $\dfrac{<\vec{S}>_{\max}}{P_{rad}/4\pi r_d^2} = \dfrac{(I_0^2\eta/4\pi^2 r_d^2)\cdot 4 \cdot 4\pi r_d^2}{27.8 \cdot I_0^2} = 4.83$

(c) Assuming at the resonance condition that $X_{ant} = 0$, the radiation resistance is given by $R_{rad} = 2P_{rad}/I_0^2$ =20.4 Ω for L = λ/2, and R_{rad} = 55.6 Ω for L = λ.

(d) To see why the full-wave dipole provides a higher radiation resistance than the half wave dipole, we first recognize that both antennas display an open-circuit condition at the ends of each dipole arm. As a result, for the half-wave antenna I_0 at the gap will correspond to the peak in the current distribution along the dipole arms. And for the full-wave antenna, I_0 will correspond to the null in the current distribution. In other words, it takes a much smaller I_0 to produce the given current distribution for the full-wave antenna than for the half-wave antenna.

Numerical Integration for Problem 4: With symbolic representation, numerical integration can be carried out with two lines in the Matlab Workspace:

(1) for L = λ/2. >> f(x) = inline('(cos(pi./2.*cos(x))./sin(x)).^2.*sin(x)')
>> y = quadl(f,0,pi)

(2) for L = λ, >> f(x) = inline('(cos(pi.*cos(x))+1)./sin(x)).^2.*sin(x)')
>> y = quadl(f,0,pi)

Note: quadl in Matlab uses the adaptive Lobatto algorithm.