HW#4 Solutions

(1) (a) the skin depth is given by $\delta = (\pi f \mu_0 \sigma)^{-1/2}$. By looking up the conductivity values, we get the skin depths shown in the following table

Material	Cu	Au	Al	Cr	Sn
Conductivity [S/m]	5.80E+07	4.50E+07	3.50E+07	7.75E+06	9.17E+06
Frequency [GHz]			Skin Depth [micron]		
10	0.661	0.750	0.851	1.808	1.662
100	0.209	0.237	0.269	0.572	0.526
1000	0.066	0.075	0.085	0.181	0.166
10000	0.021	0.024	0.027	0.057	0.053

(b) call the width of the strip a, the thickness b, and the length L. The current is confined to an annulus around the strip of thickness δ, periphery 2a + 2b, and area $\delta(2a+2b)$. Therefore the series resistance is $R_S = \rho L/A = \rho L/[\delta(2a+2b)]$, and the specific series resistance $R_S{}' = \rho/[\delta(2a+2b)] = R_S{}' = 1/\{\sigma[\delta(2a+2b)]\}$. For example, in gold (one of the most common metallizations used at THz frequencies) the skin depth from (a) is 0.075 micron and Rs' = $1.56\text{x}10^4$ Ω/m or 15.6 Ω/mm

c) For any TEM transmission line represented by the figure given, the complex propagation constant is given by $\gamma = [((R' + j\omega L')(G' + j\omega C')]^{1/2}$ where R' is the specific series resistance (same as R_S' above), L' is the specific series inductance, G' is the specific shunt conductance, and C' is the specific shunt capacitance. Here we can ignore G' since the coax will either be filled with air or some low loss plastic materials such as Teflon. Assuming R' is << ωL', (to be checked self consistently), the characteristic impedance $Z_0 = (L'/C')^{1/2}$. So for the given values, we find $L' = 2.5\text{x}10^{-7}$ H/m, which leads to the following table:

Frequency [GHz]	Rs' [Ω/mm]	ωL' [Ω/mm]	2*Real(γ) [1/mm]
100	4.9	157.1	0.10
300	8.5	471.2	0.17
600	12.1	942.5	0.24

Note that at all frequencies Rs' << ωL', so our assumptions are self-consistent. Also note that the power attenuation (absorption) per unit length, which is twice the imaginary part of γ, is modest because the units are neper/mm. So at 600 GHz, the power on this transmission line drops to exp(-0.24) = 0.78 (-1.0 dB) after one mm, or 22% of the power is absorbed. However, in 1 cm, it drops to exp(-2.4) = 0.089 (-10.5 dB) or 91% of the power is absorbed. At THz frequencies, it is important to keep transmission line lengths as short as possible !

(2) (a) the THz dielectric constants ε and associated refractive indices n of the given materials are listed in the following table. The critical angles are given by $\beta_C = \sin^{-1}(1/n)$

Material	Re ε (par)	Re ε (perp)	n (*)	theta c [deg]
Teflon	2.10	2.10	1.45	43.6
Rexolite	2.53	2.53	1.59	39.0
Mylar	2.80	2.80	1.67	36.7
Crystalline Quartz	4.34	4.27	2.07	28.9
Sapphire	11.50	9.40	3.07	19.0
high-rho Si	11.90	11.90	3.45	16.9
SI GaAs	12.80	12.80	3.58	16.2

(*) for optically anisotropic materials, the Re ε (perp) was used to compute n .

(b) from HW#3 we have an estimate of the power radiated into the substrate side of a planar dipole of physical length L,

$$P_{rad} = \frac{I_0^2 \cdot \eta}{4\pi^2} \int_0^{\pi} \cdot \int_0^{\pi} \left[\frac{\cos[(kL\cos\theta_d)/2] - \cos kL/2}{\sin\theta_d} \right]^2 \sin\theta_d \, d\theta_d \cdot d\phi \qquad (1)$$

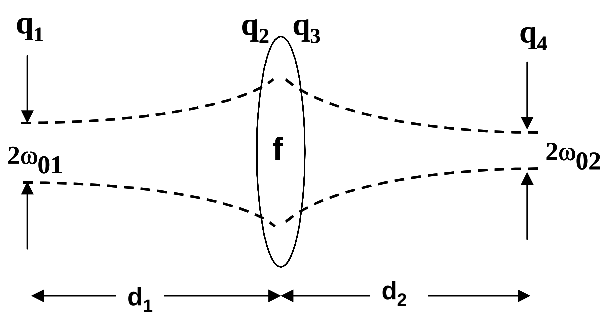
For a full wave dipole (L = λ or kL = 2π), this evaluated to $P_{rad} = 27.8 \cdot I_0^2$ on a GaAs substrate (η = 105.4 Ω). If total internal reflection is occurring, then a good approximation is to assume any radiation outside $90 - \beta_C < \phi < 90 + \beta_C$ will be totally internally reflected. Similarly, any radiation outside $90 - \beta_C < \theta < 90 + \beta_C$ will be so reflected . Since our dipole pattern has no φ dependence and kL = 2π, we can immediately write for the fraction of power that can possibly radiate out

$$P_{rad} \approx \frac{I_0^2 \cdot \eta \cdot 2\beta_c}{4\pi^2} \int_{\pi/2-\beta_c}^{\pi/2+\beta_c} \left[\frac{\cos(\pi\cos\theta_d) + 1}{\sin\theta_d} \right]^2 \sin\theta_d \, d\theta_d \qquad (2)$$

This is an easy integral to evaluate numerically and the results are shown in the following table where the power trapped is defined as the evaluation [(1) - (2)]/(1). Please note that this is just an approximation since we have not accounted for reflection coefficients and other effects.

Material	beta c	Integral	Power Trapped [%]
Air	90	27.8	0.00
Teflon	43.6	13.3	0.52
Rexolite	39	11.6	0.58
Mylar	36.7	10.8	0.61
Crystalline	28.9	7.9	0.72
Sapphire	19	4	0.86
high-rho Si	16.9	3.3	0.88
SI GaAs	16.2	3.1	0.89

(3) Focusing of THz Gaussian Beam in free space using a thin lens of focal length f.



(a) Since at the first waist R→∞, $\frac{1}{q_1} = \frac{-j\lambda}{\pi\omega_{01}^2}$; or $q_1 = \frac{j\pi w_{01}^2}{\lambda}$

Propagation through distance d_1 yields: $q_2 = q_1 + d_1 = \frac{j\pi w_{01}^2}{\lambda} + d_1$

Transformation through thin lens of focal length f yields:

$$q_3 = \frac{Aq_2 + B}{Cq_2 + D} = \frac{q_2}{\frac{-q_2}{f} + 1} = \frac{q_2 f}{f - q_2} = \frac{(\frac{j\pi\omega_{01}{}^2}{\lambda} + d_1) f}{f - j\frac{\pi\omega_{01}{}^2}{\lambda} - d_1}$$

Propagation through distance d_2 then yields:

$$q_4 = q_3 + d_2 = \frac{f(j\frac{\pi\omega_{01}{}^2}{\lambda} + d_1)}{f - d_1 - j\frac{\pi\omega_{01}{}^2}{\lambda}} + d_2$$

But at second waist $R \to \infty$, so $q_4 = j\frac{\pi\omega_{02}{}^2}{\lambda}$ (pure imaginary)

so we get: $f(\frac{j\pi\omega_{01}{}^2}{\lambda} + d_1) = (j\frac{\pi\omega_{02}{}^2}{\lambda} - d_2)(f - d_1 - j\frac{\pi\omega_{01}{}^2}{\lambda})$

which becomes: $fd_1 + j\frac{\pi\omega_{01}{}^2}{\lambda} f = (d_1 - f)d_2 + \frac{\pi^2\omega_{01}{}^2\omega_{02}{}^2}{\lambda^2} + j\frac{\pi}{\lambda}[\omega_{01}{}^2 d_2 + \omega_{02}{}^2 (f - d_1)]$

Now we can equate real and imaginary parts separately:

Real part $\Rightarrow f \cdot d_1 = (d_1 - f)d_2 + \frac{(\pi^2\omega_{01}\omega_{02})^2}{\lambda^2}$

Imaginary part $\Rightarrow \frac{\pi\omega_{01}^2(f - d_2)}{\lambda} = \frac{\pi\omega_{02}^2(f - d_1)}{\lambda}$

This is a classical pair involving 6 parameters. Often, all parameters are known (or fixed by the experiment) but two, so a unique solution for these can be found. For example, if λ, f, d_1, and ω_{01} are known, the solution for d_2 and ω_{02} become

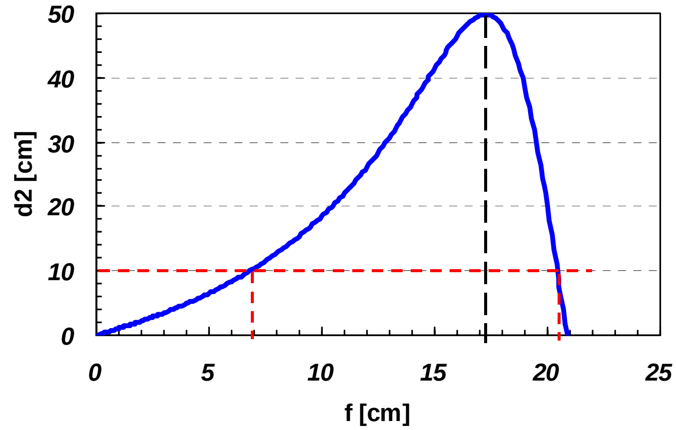
$$d_2 = \frac{f \cdot d_1(d_1 - f) + (\pi\omega_{01}^2/\lambda)^2 \cdot f}{(d_1 - f)^2 + (\pi\omega_{01}^2/\lambda)^2} \text{ and } (\omega_{02})^2 = (\omega_{01})^2 \frac{f - d_2}{f - d_1}$$

The first expression can be re-written in terms of Rayleigh length:

$$d_2 \equiv \frac{f \cdot d_1(d_1 - f) + Z_{01}^2 \cdot f}{(d_1 - f)^2 + Z_{01}^2} \,. \tag{1}$$

Logically, for a second beam waist to occur on the right side of the lens, we must have $d_2 > 0$, which means that $f \cdot d_1(d_1 - f) + (\pi\omega_{01}^2/\lambda)^2 \cdot f > 0$ or $f < d_1 + (\pi\omega_{01}^2/\lambda)^2 / d_1 = d_1 + (Z_{01})^2/d_1$

(b) Using Excel, we plot d_2 vs f under the stated conditions, as shown below. As shown in (a), there are positive solutions for d_2 when $f < d_1 + (Z_{01})^2/d_1 = 20.88$. But clearly at any f within this range, there are two possible values of f for each d_2, except right at the peak in the figure near f = 17.2 and $d_2 \approx 50$.



(c) When d_2 = 10 cm, we find from the graphical analysis above (and more accurately by numerical analysis) that f = 6.81 cm or f = 20.44 cm. Clearly f = 20.44 cm is a more practical design because it will require much less curvature on the lens and, therefore, less spherical aberration and other problems (e.g., machine tool marks or steps) that come with high curvature.

(d) Equating imaginary parts of the Gaussian-beam transformation equation yields the expression

$$\omega_{02}^2 = \omega_{01}^2 \frac{f - d_2}{f - d_1} \tag{2}$$

None of the five parameters in this equation can be negative since the focusing would not occur on the right side of the lens. *So to get the smallest possible ω_{02}, one should let d_1 approach infinity and d_2 approach f from the high side.* In this case, ω_{02} will approach zero. Although physically admissible, this is not a practical solution because as d_1 gets much larger than Z_{01}, the *spot size* of the Gaussian beam incident on the lens from the left side will grow much larger than the aperture of the lens. This causes "spillover" – a deleterious effect whereby incident radiation misses the lens entirely and, therefore, does not get focused.

(e) To avoid "spillover" d_1 is usually constrained. And d_2 is usually constrained by the size of the experiment or receiver. So one can write $d_1 + d_2 = D$. If we are free to choose f, then we can consider (2) as a function of f and differentiate ω_{02} with respect to f, subject to the constraint from part (a) $f < d_1 + (\pi\omega_{01}^2/\lambda)^2/d$. The derivative yields $\frac{\partial \omega_{02}}{\partial f} = \frac{\omega_{01}^2}{2\omega_{02}} \frac{d_2 - d_1}{(f - d_1)^2}$, which has no zeroes with respect to f (although it does have zero at $d_2 = d_1$, the so-called "confocal" solution, and a singularity at $f = d_1$), and thus has no value of f that minimizes ω_{02}. Guided intuitively by the result of part (d), we seek the smallest f possible, f_{min} and expect d_1 to be just less than D and d_2 to be very close, but not quite equal to f. A useful approximate expression for d_2 can be found by Maclaurin series expansion (freshman calculus) of (1) about f = 0 with f = f_{min}

$$d_2 = \frac{f \cdot d_1(d_1 - f) + Z_{01}^2 \cdot f}{(d_1 - f)^2 + Z_{01}^2} \approx d_2\big|_{f=0} + \frac{\partial d_2}{\partial f}\bigg|_{f=0} \cdot f_{\min} + (1/2)\frac{\partial^2 d_2}{\partial f^2}\bigg|_{f=0} f_{\min}^2 + \ldots$$

Evaluating the derivatives (the 2nd derivative being a chore), we get : $d_2\big|_{f=0} = 0...$, $\left.\dfrac{\partial d_2}{\partial f}\right|_{f=0} = 1$ and

$\left.\dfrac{\partial^2 d_2}{\partial f^2}\right|_{f=0} = \dfrac{2d_1}{d_1^2 + Z_{01}^2}$ so that we can write the "2nd order" expression $d_2 \approx f_{\min} + \dfrac{d_1}{d_1^2 + Z_{01}^2} f_{\min}^2$,

which clearly has the correct behavior as $d_1 \to \infty$, $d_2 \to f_{\min}$. Substitution of this into the (2) leads to the useful analytic approximation:

$$\omega_{02}^2 \approx \omega_{01}^2 \frac{d_1 f_{\min}^2 /(d_1^2 + Z_{01}^2)}{d_1 - f_{\min}} \tag{3}$$

This shows that a great decrease in ω_{02} compared to ω_{01} can be achieved when d_1 or Z_{01} (or both) are >> $f_{\min}$. Of course, this is dependent on the quality of the lens, which must be *aspherical* in figure to create this effect without significant aberration. A comparison of the approximation of (3) vs the exact solution from (1) and (2) is shown below for the conditions listed. The analytic form comes in handy in design work and in predicting the Gaussian-beam *magnification* M = $(\omega_{01} /\omega_{02}) \approx (d_1/f_{\min})$ for d_1 >> $f_{\min}$ and d_1 >> Z_{01} (analytics still has a place in Engineering !)