Free-Space Power Coupling for Two Special Cases: Radar and Radiometry

Friis' Transmission Formulation

Marconi was the pioneer for a new generation of electrical engineers working in the area of "wireless". One of the truly brilliant amongst these was Friis working at Bell Laboratories in the 1920s and 30s. Among other things, Friis was the first to take advantage of the inherent nature of antennas as passive, reciprocal components, and treat the free-space propagation between a transmit antenna and receive antenna as a two-port "link". This was done first and foremost for the wireless communications "link", which we review here first to set the stage for the following RF and THz sensor (i.e., radar and radiometer) "link" formulation.

The first step in Friis' formulation is the concept of an effective aperture A_{eff} for the *receiving* antenna,

$$P_{rec} = A_{eff} S_{inc}(\theta_r, \phi_r) \cdot \varepsilon_p \tag{1}$$

where P_{rec} is the power available to the antenna for delivery to a load, $S_{inc}(\theta_r, \phi_r)$ is the average Poynting vector for incoming radiation along the direction (θ_r.ϕ_r) in the spherical coordinates centered at the *receiving* antenna, and ε_p is the polarization coupling efficiency. *Note that this expression applies only when* $S_{inc}(\theta_r, \phi_r)$ *is aligned with the direction of the beam-pattern maximum*. When there is misalignment, another factor is required which is the just the receive beam-pattern,

$$P_{rec} = A_{eff} \cdot F_r(\theta_r, \phi_r) S_{inc}(\theta_r, \phi_r) \cdot \varepsilon_p \ . \tag{2}$$

Next, we suppose that this *received* Poynting vector is generated by a second, *transmitting* antenna. We can relate the received power to the properties of the transmitting antenna by:

$$S_{inc}(\theta_r, \phi_r) \equiv S_t(r, \theta_t, \phi_t) = \frac{P_{rad} D_t \cdot F_t(\theta_t, \phi_t)}{4\pi r^2} \tau(r) \equiv \frac{P_{in} G_t \cdot F_t(\theta_t, \phi_t)}{4\pi r^2} \tau(r) \tag{3}$$

where the subscript "t" is for transmitting, P_{rad} is the total radiated power, P_{in} is the power used to drive the transmitting antenna (in the matched case, equal to P_{rad}), θ_t and ϕ_t are the spherical angles in the spherical coordinate system centered at the transmitting antenna, τ (r) is the power transmission function including all attenuation effects, and r is the distance (i.e., the "range" between transmitter and receiver. In writing (3) it is understood that F_t is taken in the direction (θ_t,ϕ_t) pointing towards the receiver, which is not necessarily the direction of the maximum of F_t. Substitution of (3) into (2) yields the relationship

$$P_{rec} = A_{eff} \frac{P_{in} G_t \cdot F_r(\theta_t,\phi_t) F_t(\theta_r,\phi_r)}{4\pi r^2} \tau(r) \cdot \varepsilon_p \tag{4}$$

This can be simplified further in terms of the (ostensibly) known parameters of the receiving antenna using the relationships,

$$\mathrm{P_{out}} = \mathrm{P_{rec}} \frac{\mathrm{G_r}}{\mathrm{D_r}} \equiv \mathrm{P_{rec}} \frac{\mathrm{G_r}}{4\pi \mathrm{A_{eff}} / \lambda^2} \tag{5}$$

where P_{out} is the power delivered to the load of the receiving antenna. Substitution of (4) into (5) yields

$$P_{out} = P_{in} \left(\frac{\lambda}{4\pi r_1} \right)^2 G_r G_t \cdot F_r(\theta_r,\phi_r) F_t(\theta_t,\phi_t) \cdot \tau(r) \cdot \varepsilon_p \tag{6}$$

the expression commonly known as Friis' formula after its originator. It effectively treats the antenna combination like a two-port circuit with the pattern angular dependence and polarization dependence included explicitly. The term $(\lambda/4\pi r)^2$ is called the free-space loss factor, which is of considerable practical and historical importance. Several theoreticians of the 19th century believed that radiation would decay faster than $1/r^2$ from a source. It was Hertz's observation of this $1/r^2$ dependence of radiation that encouraged the technology of "wireless."

Friis' transmission for Radar

For radar systems, the transmitter (in systems engineering often shortened to "Tx") and the receiver (often shortened to "Rx") have, in addition to free space, a body between them (i.e., the radar "target") that scatters electromagnetic radiation from the Tx to the Rx. To first order, some bodies (particularly round metallic ones) absorb practically none of the incident power and, instead, scatter it isotropically. Conceptually, we can then think of the body as a passive Rx/Tx combination that receives a power according to (1) above and transmits it isotropically, so that

$$P_{inc} = A_{eff} \, |\vec{\bar{S}}_{inc}(\theta,\phi)| \equiv \sigma \, |\vec{\bar{S}}_{inc}(\theta,\phi)|, \qquad \text{and} \tag{7}$$

$$|\vec{\bar{S}}_{scatt}(\theta,\phi)| = P_{inc} / 4\pi r_2^2 , \tag{8}$$

where σ is the (target) scattering cross section and r_2 is the distance between the scatterer and the observation point. We now assume that S_{inc} originates from a Tx antenna and S_{scatt} radiates back to a second (Rx) antenna to create an "echo" of received-aperture power P_{rec} . In this case,

$$|\vec{\bar{S}}_{inc}(\theta,\phi)| = \frac{P_{in} G_t \cdot F_t(\theta_t,\phi_t)}{4\pi r_1^2} \tau(r_1) \qquad \text{and} \tag{9}$$

$$P_{rec} = A_{eff} \, F_r(\theta_r,\phi_r) \, |\vec{\bar{S}}_{scatt}(\theta,\phi)| \, \varepsilon_p \tag{10}$$

where r_1 is the distance between Tx and the scatterer, and ε_p is the fraction of the scattered power that has the same polarization characteristics as the Rx antenna. As in (5) above, we assume to know the Rx properties so that

$$P_{out} = P_{rec} \frac{G_r}{D_r} \equiv P_{rec} \frac{G_r}{4\pi A_{eff} / \lambda^2} \tag{11}$$

where P_{out} is the power delivered from the Rx antenna to its load. By substitution of (9) into (7), (7) into (8), (8) into (10), and (10) into (11), we find the relation

$$P_{out} = \tau(r_1)\tau(r_2)P_{in}\frac{\sigma\lambda^2}{(4\pi)^3 r_1^2 r_2^2}G_r G_t \cdot F_r(\theta_r,\phi_r)F_t(\theta_t,\phi_t)\cdot\varepsilon_p \qquad (12)$$

This is the famous "bistatic" (two stationary point) radar transmission equation. In the special ("monostatic") case that the transmitter and receiver share a common antenna, $r_1 = r_2$, $G_r = G_t$, $F_r = F_t$, $\tau(r_1) = \tau(r_2) = \tau(r)$, and (12) reduces to

$$P_{out} = [\tau(r)]^2 P_{in}\frac{\sigma}{4\pi r^2}\cdot\left(\frac{\lambda}{4\pi r}\right)^2 G^2 \cdot [F(\theta,\phi)]^2\cdot\varepsilon_P \qquad (13)$$

Like Friis' formula for communications, this treats the radar problem like a two-port equivalent circuit. But physically it differs from Friis' with the additional r^{-2} factor, leading to an overall r^{-4} dependence of P_{out} on P_{in}. This result is of great practical importance because it generally means that radar systems must transmit much higher power levels than communications systems to achieve a satisfactory received power for signal processing.

Example: One application for THz radar systems is short-range concealed object detection and imaging. This example calculates (a) the received power and (b) the background-limited signal-to-noise ratio (SNR) for a 600 GHz bistatic coherent radar located 1 m from the target.
(a) To get the received power, we make the following practical assumptions (1) the transmit and receive feedhorns are located side-by-side in close enough proximity that their antenna patterns in space overlap perfectly, (2) the transmit power is P_{in} = 1 mW at 600 GHz (λ = 0.5 mm), a very small number by microwave radar standards, but about all that is available today from affordable solid-state sources at 650 GHz (centered at one of the THz "windows"), (3) the antenna gain is 100 (20-dB), easy to achieve from "standard gain" feedhorns without any other optics, (4) the target RCS is $\sigma = 10^{-4}$ m^2 (e.g., barrel of a hand gun), and (5) the polarization is scrambled so that ε_p = 0.5 (most concealed objects of interest, such as guns, have complicated shapes so are bound to scramble the incident polarization upon scattering). The maximum antenna output power occurs when the peak of the antenna pattern is aimed at the target (i.e., $F(\theta,\phi)$ = 1.0). If we assume zero attenuation between Tx and the scatterer, we get $P_{out} = 5.4\text{x}10^{-14}$ W, down over 10 orders of magnitude compared to the transmit power ! And this is the "best-case scenario" since in the THz region the atmosphere and the materials concealing the object always attenuate significantly. While appearing hopelessly low, this level of received power is not unusual in radar systems and remote sensors of all types, and can lead to useful detection if the receiver is designed correctly.
(b) The background-limited SNR can be calculated assuming a terrestrial environment, single spatial mode operation of the receive feedhorn, and "bandlimited" receiver of bandwidth Δf. In this case, the average thermal power and rms power fluctuation collected by the receiver is just $P_N = k_B T \Delta f$ where the background temperature is generally ~300 K anywhere in the THz region. One advantage of a coherent radar over an incoherent one (more on this later) is that the bandwidth can be made much more narrow, just great enough to accommodate the modulation bandwidth of the coherent "waveform" from the transmitter. We will address "waveforms" more later on, but suffice it to say that modulation is generally used, sometimes amplitude modulation, sometimes phase or frequency modulation, to get other information from the radar about the target, such as its range. Another reason to modulate the transmit power is to help mitigate the effect of other scatterers within the field-of-view of the radar, that collectively get called "clutter". For the present purpose, let's assume the receiver bandwidth is 1 MHz. The rms noise power is then $P_N = 4.1\text{x}10^{-15}$ W, leading to a "background-limited" signal-to-noise ratio of 13 or 11 dB. As we will see later, insertion losses and physical noise in the receiver will generally make the background-limited SNR significantly lower than this, and clutter makes it worse yet. But the background-limited value forms a "best-case scenario" this is always good to know in systems engineering.

The radar-transmission equation is often re-formulated because the Rx output power is often known relative to a lower-limit, P_{out}^{min}, dictated by noise in the receiver electronics or from environmental effects. Then the interesting question is the maximum range at which a target of cross section σ can be measured. Re-arranging Eq. (13) and assuming τ is independent of r we find

$$r = \left(\frac{P_{in}}{P_{out}} \cdot \frac{\sigma\lambda^2}{(4\pi)^3} \tau G^2 \cdot [F(\theta,\phi)]^2 \cdot \varepsilon_p \right)^{1/4} \tag{14}$$

which is one form of the "radar range equation" - very useful expression for predicting the performance of radar systems under ideal conditions. In many practical cases, r is limited to a maximum possible value by the fact that P_{out} is masked by receiver noise, as shown next.

Another important application of (13) is the minimum transmit power required to achieve a certain minimum receiver power P_{min}. This can be calculated as

$$P_{in} = P_{\min} \left\{ \tau(r)]^2 \frac{\sigma}{4\pi r^2} \cdot \left(\frac{\lambda}{4\pi r} \right)^2 G^2 \cdot [F(\theta,\phi)]^2 \cdot \varepsilon_P \right\}^{-1} \tag{15}$$

In some literature this is called the minimum required transmit power.

Received Power in Radiometry: Antenna Back-Projection

Antenna reciprocity is a very important concept in RF sensors since it allows us to think about antennas interchangeably in transmit or receive modes knowing just the radiation pattern and its properties. In radiometry we are usually concerned with detecting thermal radiation emitted from a target, or environmental radiation reflected from a target, at a distance great enough that the angle subtended by the target at the sensor, Ω_T, may be greater or less than the diffraction-limited beam able Ω_B.

An interesting application of the antenna theorem comes in using Planck's radiation law to estimate the portion of the thermal emission from targets received by (passive) radiometers. We start by re-writing the blackbody spectral density function, this time for just one polarization and one direction (hence the 4x reduction compared to that derived earlier) since different polarizations constitute different spatial modes:

$$\frac{dP}{d\nu} = \frac{A \cdot 2\pi h\nu^3}{c^2 (e^{h\nu/k_B T} - 1)} = \frac{2\pi A}{\lambda^2} \frac{h\nu}{(e^{h\nu/k_B T} - 1)}$$

And now our interpretation of A is the antenna effective aperture. Application of the diffraction-limited antenna theorem yields

$$\frac{dP}{d\nu} = \frac{2\pi}{\Omega_B} \frac{h\nu}{(e^{h\nu/k_B T} - 1)}$$

We recognize the factor $2\pi/\Omega_B$ as the number of spatial modes contained in one hemisphere if decomposed into the diffraction-limited solid angle Ω_B of the antenna.

Rarely, if ever, does a target occupy 2π steradians in space with respect to the sensor. And generally the antennas in RF sensors operate with just one spatial mode. So to estimate the received power, we use reciprocity to *back-project* the diffraction-limited beam to the target, recognizing that the number of diffraction-limited solid angles filled by the target is one if $\Omega_T > \Omega_B$, but less than one of $\Omega_T < \Omega_B$. Mathematically, this leads to the received power spectral density:

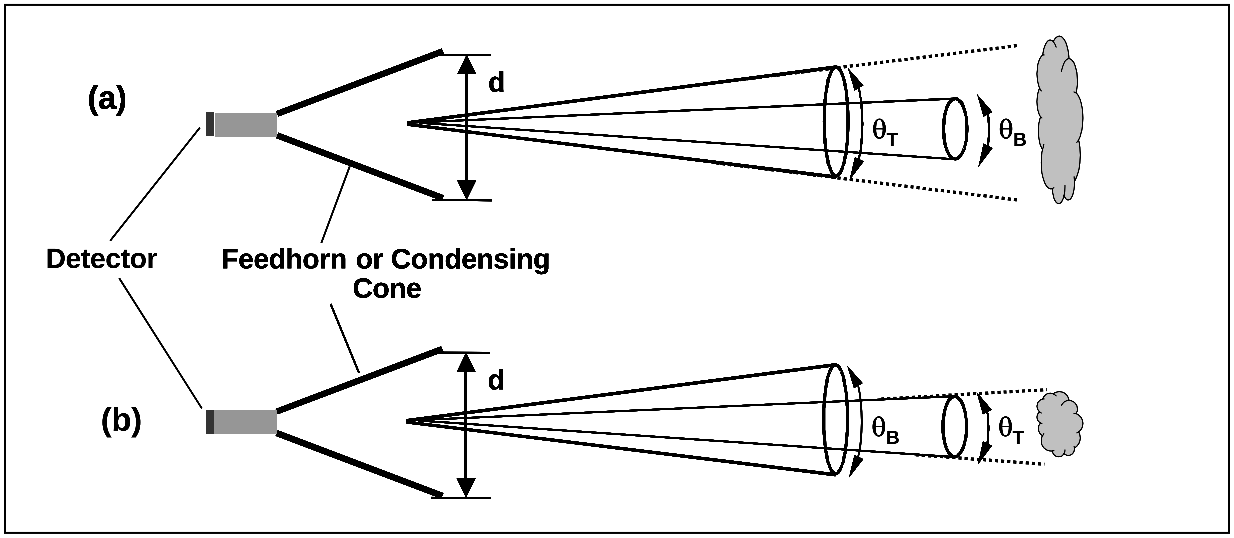
$$\frac{dP}{d\nu} \approx \left[\frac{\Omega_T}{\Omega_B} \Theta[\Omega_B - \Omega_T] + \Theta[\Omega_T - \Omega_B] \right] \frac{h\nu}{(e^{h\nu/k_B T} - 1)}$$

where Θ is the Heaviside unit step function. This has the expected behavior that if the angle subtended by the target is larger than the beam solid angle and in the Rayleigh-Jeans limit,

$$\frac{dP}{d\nu} \approx k_B T$$

Sometimes this is called the "overfilled" condition. But if the target angle is smaller than the beam solid angle, then the received power is less than expected from thermodynamic equipartition by the spatial "fill factor", Ω_T/Ω_B,

$$\frac{dP}{d\nu} \approx \left[\frac{\Omega_T}{\Omega_B} \right] \frac{h\nu}{(e^{h\nu/k_B T} - 1)} \rightarrow \frac{\Omega_T}{\Omega_B} k_B T$$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