Gaussian-beam methodology

A key assumption behind the predictions given from scalar diffraction theory is that the illumination across the aperture is uniform. This is a good assumption in some circumstances such as predicting the power collected by a receive antenna from a distance source whose pattern beam-width measured at the receive antenna is much larger than the lateral extent of the receive antenna. But there are other times when the uniform-intensity assumption is inaccurate, such as in describing the radiation coupled to a fundamental-mode feedhorn or planar antenna by a second feedhorn or antenna a short distance away. This situation arises in many THz transceivers and simple bench-top set-ups where the power is transferred in free space "quasi-optically" from component-to-component; i.e, using traditional optical components from the visible region of the spectrum, such as lenses, but at the much longer wavelengths of THz or even lower-frequencies. As a matter of fact, such "quasi-optical" techniques started early in the history of microwave and millimeter-wave systems, but have been gradually replaced by transmission-line or guided-wave coupling techniques as printed integrated circuits have become available. However, quasi-optical techniques persist in the THz region, largely because integrated circuits are much less prevalent, and the transmission lines and waveguides that interconnect them have much greater attenuation than at lower frquencies.

One of the useful features of scalar-diffraction theory is its ability to predict what happens to the radiation once the uniform-illumination assumption is violated. For example, if the aperture is circular and if the illumination distribution is a Gaussian in the lateral plane with respect to the axis of symmetry, then the radiation pattern is also Gaussian, at least in the far-field limit. Intuitively, this makes sense since in this limit the Fresnel-Kirchoff integral reduces to a Fourier transform, and the Fourier transform of a Gaussian is always a Gaussian. It turns out that this result, commonly known as the Gaussian beam pattern, also applies to the near-field behavior with increasing accuracy as d/λ increases far beyond unity.

Although the Gaussian-beam result was known early in the history of electromagnetics, it was apparently not fully appreciated until the advent of the laser. The gain media in gas and solid-state lasers typically have very large values of d/λ, and generally emit much greater intensity at the center than at the lateral edges. A useful way to develop the Gaussian behavior is to model the gain medium with a quadratic complex refractive-index lateral profile.[1]

In the mm-wave and THz region, the applicability of Gaussian beams is less obvious on first glance, but becomes plausible when one considers the coupling between antennas and

[1] A. Yariv, "Quantum Electronics, 2nd Edition", (Wiley, New York, 1975), Chapter 6.

circuits. MM-wave and THz antennas are often operated in their fundamental spatial modes for which the radiation intensity is maximum but rather slowly varying along the propagation axis. Then at the characteristic angle $\theta \approx \beta/2$ away from the axis the radiation begins falling rapidly in the lateral directions, with some radiation inevitably occurring at larger angles in the sidelobes because of diffraction. All these properties except the undulation of the sidelobes are described rather well by a Gaussian function vs r with perfect azimuthal symmetry about the z axis. This is the so-called fundamental Gaussian mode. Other possible symmetric functions, such as a sech^2 or Lorentzian, are either too steep about the propagation axis or decay too slowly at large angles.

MM-wave and THz antennas are often operated in fundamental mode for practical reasons. One reason is that the fundamental mode is generally the most symmetric and has the smallest beam width of all possible antenna modes. Another reason is that the devices and circuits to which the antenna is coupled are designed for their own fundamental mode, be it in high-frequency transmission line or waveguide. This is usually the easiest and most effective way to design mm-wave and THz active devices and circuits, but it generally makes the coupling between the circuits and antenna efficient only for one antenna mode, usually the fundamental mode. These considerations break down, of course, if none of the electronics or components coupled to the antenna need to process radiation at high frequencies. Such is the case, for example, in mm-wave and THz bolometers which merely rectify any incident power absorbed. Hence, bolometers can be and often are mounted in multimode antenna-like structures, such as integrating cavities, to maximize the sensitivity and spectral bandwidth.

Finally, it is important to realize that any Gaussian beam is just one of an infinite number of modes forming an orthonormal TEM basis set. This basis set is as applicable to representing arbitrary radiation in free space as any other orthonormal basis, and must comply with the antenna theorem defined earlier. Each function of the basis set differs from the others in the degree of azimuthal symmetry about the propagation direction. As alluded to above, the fundamental or TEM_{00} mode is the only one with perfect azimuthal symmetry, and is therefore the most popular and useful in solving free-space propagation problems in THz systems.

Formulation

One benefit of the Gaussian-beam approach is it tends to add improved accuracy in mm-wave and THz design with only a minor increase in difficulty. This is because Gaussian beams, much like Gaussian distributions in probability theory, behave well mathematically under system-level operations. Given a propagation direction along the z axis, the fundamental TEM_{00} Gaussian beam is given in cylindrical coordinates (r,θ,z) by

$$E(r,z) = E_0 \frac{\omega_0}{\omega(z)} \exp[-r^2/\omega^2] \cdot \exp[-j(kz - \phi + kr^2/2R)] \qquad (1)$$

where E_0 is the maximum electric field amplitude, η is the intrinsic impedance of the medium of propagation, ϕ is a phase constant, ω is the radius where the intensity drops by e^1 relative to the on-axis intensity (i.e., the "spot size"), or "beam waist," and R is the radius of curvature. Fortunately, all of these quantities are inter-related through simple algebraic expressions:

$$\omega^2(z) = \omega_0^2 \cdot \left[1 + \left(\frac{\lambda \cdot z}{\pi \cdot \omega_0^2}\right)^2\right]$$

$$R(z) = z \cdot \left[1 + \left(\frac{\pi \cdot \omega_0^2}{\lambda \cdot z}\right)^2\right] \qquad (2)$$

$$\phi(z) \ = \tan^{-1}\left(\frac{\lambda z}{\pi \omega_0^2 n}\right)$$

from these expressions it is clear that ω(z) ≥ ω_0, so that ω_0 is the minimum spot radius and the maximum electric field occurs in the plane of constant z where ω(z) = ω_0 which defines the "beam waist." In this plane the following simple form of the electromagnetic intensity is valid:

$$I(r) = \frac{E_0^2}{2\eta} \cdot \exp[-2r^2/\omega_0^2]$$

where η is the intrinsic impedance of the propagation medium. Because the Gaussian integral is analytic if taken from r = 0 to ∞, the following useful relationship exists between the total propagating power and on-axis intensity at the beam waist:

$$I(r=0) = P \cdot \left(\frac{2}{\pi \cdot \omega_0^2}\right)$$

Further satisfying properties of the Gaussian beam follow from Eqns 1 and 2 in the limit of positive z $>> \frac{\pi \cdot \omega_0^2}{\lambda} \equiv z_0$. In this case, R(z) → z , $\omega^2 \rightarrow \left(\frac{\lambda \cdot z}{\pi \cdot \omega_0}\right)^2 = \left(\frac{\lambda \cdot R}{\pi \cdot \omega_0}\right)^2$, θ → π/2, and the complex exponential approaches $j\exp[-jkz(1+r^2/2z^2)]$. Hence, provided that the observation point is such that r < z or R, the resulting electric field behaves as

$$E(r,z) \rightarrow E_0 \frac{j\pi\omega_0^2}{\lambda R} \exp[-(\pi\omega_0 r^2/(\lambda R)^2] \cdot \exp[-j(kR)]$$

which is the expected form of a spherical wave weighted by a beam-pattern function. The important distance parameter z_0 is called the Rayleigh length. It also happens to be the distance for which R(z) equals its minimum value.

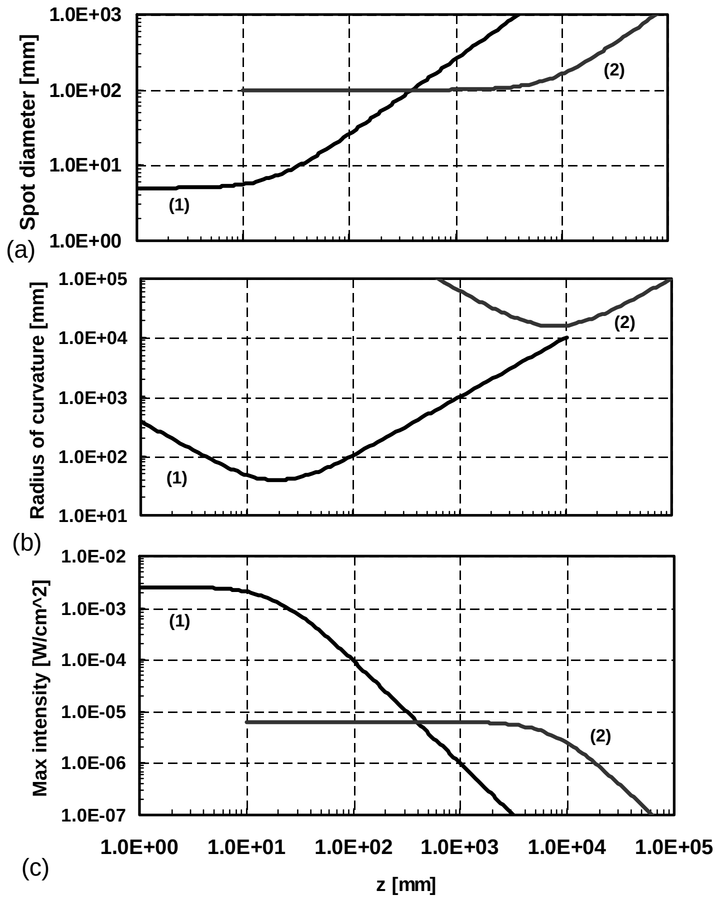

Fig. 1. Gaussian parameters: (a) spot diameter ($2\cdot\omega_0$) (b) radius of curvature, and (c) on-axis intensity for two typical mm-wave or THz situations: (1) λ = 1 mm, P = 1 mW, minimum spot size at beam waist = 2.5 mm; (2) λ = 1 mm, P = 1 mW, minimum spot size at beam waist = 50 mm.

Another satisfying property is found by tracking the r = ω or 1/e profile of the beam in the "far-field" limit z >> z_0. In this case the r = ω locus asymptotically approaches a full angle (centered about the z axis) of

$$\theta_B = 2\cdot\tan^{-1}\left(\frac{\omega}{z}\right) \rightarrow 2\cdot\tan^{-1}\frac{\lambda}{\pi\cdot\omega_0} \approx \frac{2\cdot\lambda}{\pi\cdot\omega_0}$$

Re-defining ω_0 as the lateral extent d of the Gaussian beam at its minimum aperture, we see once again a dependence of the far-field behavior on the ubiquitous "diffraction" ratio, λ/d.

All of these properties are exemplified in the curves shown in Fig. 1 for two representative Gaussian beams propagating in free space, one with a minimum spot diameter $2\omega_0$ = 5 mm (0.2 inch) and the other with $2\omega_0$ = 100 mm (4 inch). The wavelength of both is 1 mm and beam waists occur at z = 0. For the smaller-waist beam, the Rayleigh length z_0 is only ≈ 20 mm so the beam quickly diverges to a full divergence angle of about 14^{o}. For the larger-waist beam, z_0 is about 8 m, so the beam remains highly collimated out to this distance and then begins to approach a divergence angle of just under 1^{o}. Note that this larger beam would not be too difficult to support by a relatively simple telescope of aperture about 8 inches or more in diameter – that is, a man-portable instrument.

The behavior of the larger beam in Fig. 1 illustrates an important potential advantage of mm-wave and THz propagation over that in the lower RF bands. Namely, in applications where the remote sensor supports a Gaussian beam and the object or target is at a *short range* not much greater than the Rayleigh length, the divergence of the beam between the two can be very small. And thus the intensity will drop far slower than the $1/r^2$ spherical-wave behavior in the "far-field" of every common antenna.

Transformation of Gaussian beams: An representative system example

A second benefit of the Gaussian beam approach is its tendency to remain Gaussian through transformation by various optical two-port components, such as lenses and mirrors. As in microwave network theory, passive optical two-ports can be represented by a number of different 2x2 matrix formulations depending on the physical formulation of the propagating electromagnetic mode. A common formulation in optics is the "ray," represented by a column or row vector [r(z), r'(z)], where r is the distance from the propagation axis and r' is the slope of the ray with respect to this axis. Optical components are represented by 2x2 ABCD matrices, and an input ray is transformed according to

$$\begin{pmatrix} r_{out} \\ r_{out}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_{in} \\ r_{in}' \end{pmatrix}$$

Some good examples of such matrices are: (1) free space path of length L, A=1,C = 0, and D = 1; (2) thin lens of focal length f: A = 1, B = 0, C = -1/f, D = 1. The accuracy of this formulation is best for "paraxial" rays, i.e., those propagating close to the optical axis.

Remarkably, the ABCD representation also applies to Gaussian beam propagation through the definition of a complex Gaussian beam parameter

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{n\pi\omega^2}$$

The transformation equation is then given by

$$q_{i+1} = \frac{Aq_i + B}{Cq_i + D}$$

where q_i is the beam parameter in a plane $z = z_i$. Note that the free-space ABCD matrix simply transforms as $q_{i+1} = q_i + L$.

As an illustrative example of the Gaussian beam approach, Fig. 2 shows the modeling of the radiation propagation in a system familiar to the author: the THz photomixing spectrometer. In this case, coherent mm-wave and THz radiation is generated selectively by beating two frequency-offset lasers in a small photoconductive element mounted at the driving point of a planar antenna. The planar antenna is located on a semi-insulating GaAs or InP substrate – both having very low absorption in the THz region but difficult to make antennas with because of their high dielectric constant , $\varepsilon_r \approx 13$. Therefore, the photomixer substrate is abutted to the back-side of a high-resistivity silicon hyperhemisphere. The radiation coming out of the hyperhemisphere will likely be diverging, so a second focusing lens (e.g., plastic) is added at some distance to focus the radiation down to a beam waist. Since the purpose of the spectrometer is to provide radiation to a sample cell for THz spectroscopic analysis, an interesting question is if and where this beam waist will occur, and how big the minimum spot size will be.

The solution is found by first estimating the pattern coming out of the planar antenna as an equivalent Gaussian beam as shown in the exploded view of Fig. 2. The two curved loci in this view represented the $r = \omega$ points. The beam propagates through the GaAs or InP substrate, into the Si hyperhemisphere, and is then transformed into free space using an ABCD matrix appropriate to a spherical-dielectric interface.[2] To avoid significant total-internal reflection, the hyperhemisphere can provide only a slight transformation of the beam, which remains diverging after passage through the Si-air interface.

The free-space Gaussian beam then propagates to the (plastic) plano-convex lens which transforms the diverging Gaussian beam to a converging beam through the application of the thin-lens ABCD matrix. By judiciously varying the hyperhemisphere-to-plano-convex separation, we achieve the beam waist shown in Fig. 2. The waist shown has a minimum spot size $\omega_0 \approx 1.5$ mm, consistent with the given frequency (300 GHz), the planar-antenna spot size (0.2 mm), the radius and set-back of the Si hyperhemisphere (5.0 and 1.76 cm, respectively), and the focal length and diameter of the plastic lens (2.0 and 2.0 inch, respectively).

[2] A. Yariv, IBID.

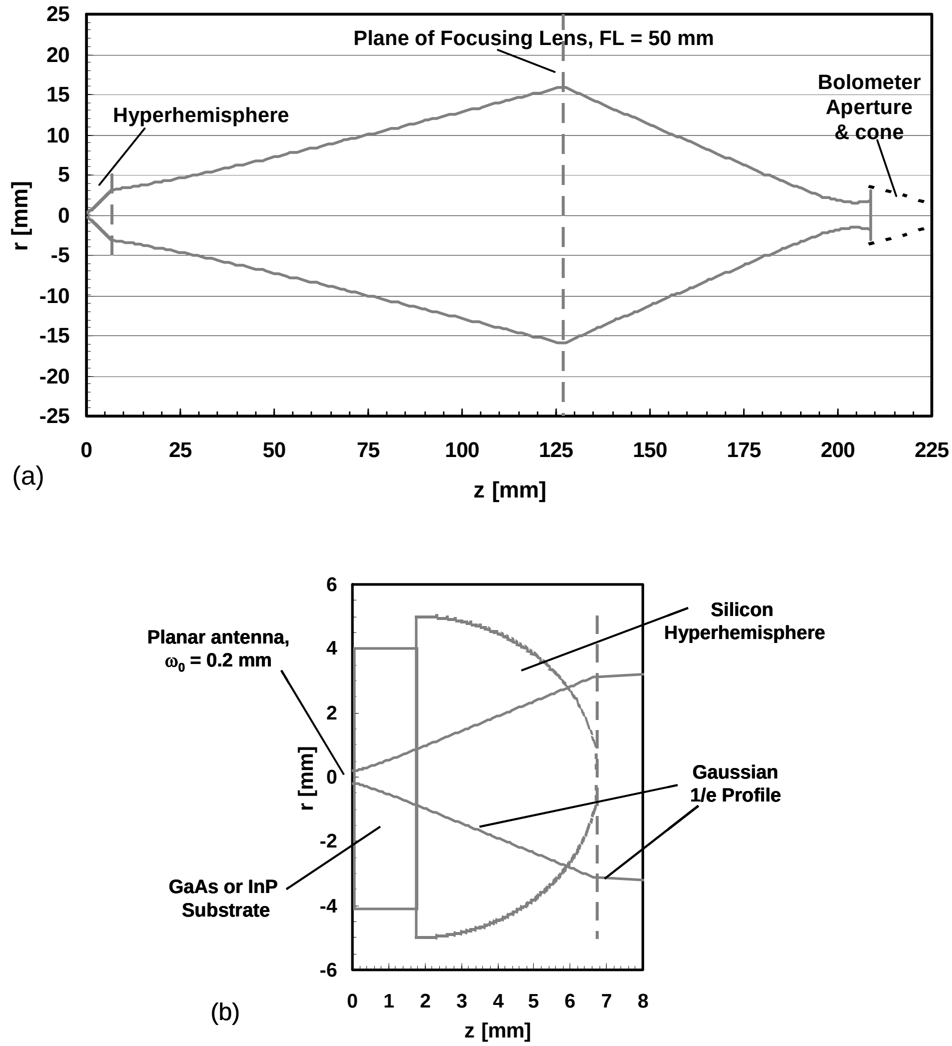

Fig. 2. (a) Results of Gaussian-beam design of THz photomixer spectrometer as a representative problem in system-level free-space radiation transformation and coupling. (b) Exploded view of photomixer region consisting of planar antenna, semi-insulating substrate and Si hyperhemisphere.